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AN ELEMENTARY GEOMETRY

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WITH NUMEROUS EXERCISES ILLUSTRATIVE OF THE PRINCIPLES OF EACH BOOK.

University Edition.

BY

WILLIAM F. BRADBURY, A.M.,

HOPKINS MASTER IN THE CAMBRIDGE HIGH SCHOOL; AUTHOR OF AN ELEMENTARY ALGEBRA, AN ELEMENTARY GEOMETRY AND TRIGONOMETRY, AND A TREATISE ON TRIGONOMETRY AND SURVEYING.

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PREFACE.

The favor with which the author's smaller work on Elementary Geometry has been received has induced him to undertake the present more complete work, in the hope that it may prove equally useful to the higher classes of learners for whom it is intended.

While each Book has been made fuller, the same plan has, for the most part, been followed as in the former work: as in that, numerous practical questions illustrative of each Book, and theorems for original demonstration are introduced, serving as practical applications of the principles of the Book, and for discipline in discovering methods of demonstration. In addition to the exercises at the end of each Book many more, arranged in proper order, have been added at the close of the whole. These features are believed to be of special value in securing a real acquaintance with Geometry and its practical application.

In the discussion on the area of the rectangle and the circle, and the volume of the rectangular parallelopiped and the sphere, a method different from that in the smaller work has been adopted as better for the class of learners for whom this work is designed. The *direct* method of proof has been used in propositions usually proved by the indirect (see 85, last part of 87, and 102, in Book I.).

In the preparation of this work the author has obtained valuable suggestions from many European works on Elementary Geometry, and especially from the French works of Montferrier and of Rouché and Comberousse.

Of the points in which the author claims special originality, attention is called to Propositions XVIII. (including its Corollaries) and XX. of Book I.; the definition and consequent discussion of Similar Polygons (II. 52 – 58, 76 – 78); the use made of Proposition X., of Book III., in subsequent demonstrations; and the definition and consequent discussion of Similar Solids (VII. 78 – 82).

For the introduction of the terms "Normal to a Plane," and "Aspect of a Plane," the author is indebted to James Mills Peirce, Professor of Mathematics in Harvard University. By the use of these terms the author is enabled to extend to planes the same idea as is used in the definition and treatment of lines and of angles in Book I. For a discussion of the word "Aspect," as applied to planes, those interested are referred to several articles in the London journal, "Nature," for the years 1871 – 72, and specially to an article, by Professor J. M. Peirce, on p. 102, Vol. V., of the same journal.

W. F. B.

Cambridge, Mass., April, 1877.

CONTENTS.

GEOMETRY.

BOOK VII.

BOOK VIII.

BOOK IX.

EXERCISES.

ELEMENTARY GEOMETRY.

INTRODUCTORY DEFINITIONS.

1. Mathematics is the science of quantity.

2. Quantity is that which can be measured; as distance, time, weight.

3. Geometry is that branch of mathematics which treats of the properties of extension.

4. Extension has one or more of the three dimensions, length, breadth, or thickness.

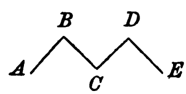
5. A **Point** has position, but not magnitude.

6. A **Line** has length, without breadth or thickness.

7. A **Straight Line** is one whose direction is the same throughout; as *A B*.

A straight line has two directions exactly opposite, of which either may be assumed as its direction.

8. A **Broken Line** is a continuous line formed of different straight lines; as *A B C D E*.



9. A **Curved Line** is one whose direction is constantly changing; as *C D*.

10. A **Surface** has length and breadth, but no thickness.

11. A **Plane** is such a surface that a straight line joining any two of its points is wholly in the surface.

12. A **Solid** has length, breadth, and thickness.

13. *Scholium.* The boundaries of solids are surfaces; of surfaces, lines; the ends of lines are points.

14. A **Theorem** is something to be proved.

15. A **Problem** is something to be done.

16. A **Proposition** is either a theorem or a problem.

17. A **Corollary** is an inference from a proposition or statement.

18. A **Scholium** is a remark appended to a proposition.

19. An **Hypothesis** is a supposition in the statement of a proposition, or in the course of a demonstration.

20. An **Axiom** is a self-evident truth.

AXIOMS.

21. If equals are added to equals, the sums are equal.

22. If equals are subtracted from equals, the remainders are equal.

23. If equals are multiplied by equals, the products are equal.

24. If equals are divided by equals, the quotients are equal.

25. Like powers and like roots of equals are equal.

26. The whole of a magnitude is greater than any of its parts.

27. The whole of a magnitude is equal to the sum of all its parts.

28. Magnitudes respectively equal to the same magnitude are equal to each other.

29. A straight line is the shortest distance between two points.

PLANE GEOMETRY.

BOOK I.

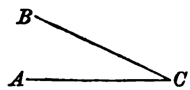
POINTS, LINES, ANGLES, POLYGONS.

DEFINITIONS.

30. PLANE GEOMETRY treats of figures whose elements are all in the same plane.

THE POINT.

31. The position of a point is determined by its distance and direction from a known point; or by its direction from two known points, provided the three points are not all in the same straight line. Thus, the position of the point C is known, if the distance from the known point A, and the direction of C from A are known; that is, if the length and direction of AC are known. The position of the point C is also known if its direction from two known points, A and B, is known, provided A, B, and C are not in the same straight line; for C, being in the lines AC and BC, must be at their point of intersection.



THE STRAIGHT LINE.

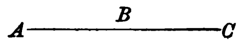
32. As a straight line has the same direction throughout (7),* two points, or one point and the direction, of a straight line determine its position.

* The figures alone in parentheses refer to an article in the same Book; in referring to an article in another Book, the number of the Book is prefixed. For convenience, the Introductory Definitions are numbered as though a part of Book I.

33. A straight line being the shortest distance between two points (29) is considered *the distance* between the points.

The word *line*, used alone hereafter, means a straight line.

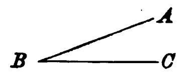
34. The *origin of a line* is the point at which the line is supposed to begin; or *from* which it is produced. Thus the line $A\ B$ is produced if it is extended toward C, A being considered the origin; but $C\ B$ is produced, if it is extended toward A, C being considered the origin.



ANGLES.

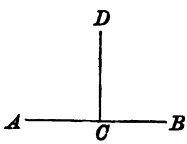
35. An **Angle** is the difference in direction of two lines.

If the lines meet, the point of meeting, B, is called the *vertex*; and the lines $A\ B$, $B\ C$, the *sides* of the angle.



If there is but one angle, it can be designated by the letter at its vertex, as the angle B; but when a number of angles have the same vertex, each angle is designated by three letters, the middle letter showing the vertex, and the other two with the middle letter the sides; as the angle $A\ B\ C$.

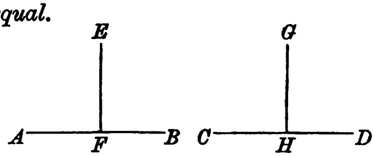
36. If a straight line meets another so as to make the adjacent angles equal, each of these angles is a *right angle;* and the two lines are perpendicular to each other. Thus, $A\ C\ D$ and $D\ C\ B$, being equal, are right angles, and $A B$ and $D C$ are perpendicular to each other.



THEOREM I.

37. *All right angles are equal.*

Let $E F$ and $G H$ be perpendicular respectively to the straight lines $A\ B$ and $C\ D$; the right angles $A\ F\ E$, $C\ H\ G$, $E\ F\ B$, $G\ H\ D$, are equal.



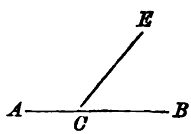
Place the line $A B$ on $C D$; $A B$ will coincide with $C D$ (32); therefore if F be considered the origin of the lines $F A$, $F B$, and H, of $H C$, $H D$, the difference of direction of $F A$ and $F B$ is equal to the difference of direction of $H C$ and $H D$; that is, the angle formed at F by $F A$ and $F B$ is equal to the angle formed at H by $H C$ and $H D$; and as these equal angles are respectively bisected by the perpendiculars $E F$, $G H$, the angles $A F E$, $C H G$, $E F B$, $G H D$, must be equal (28).

38. *Scholium.* In the measurement of angles, the right angle, being an invariable quantity, is often taken as unity.

39. *Corollary.* From a given point in a straight line but one perpendicular can be drawn in the same plane.

DEFINITIONS.

40. An **Acute Angle** is less than a right angle; as $E C B$.



41. An **Obtuse Angle** is greater than a right angle; as $A C E$.

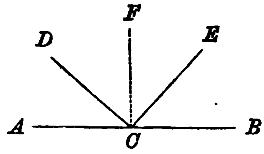
Acute and obtuse angles are called oblique angles.

42. The **Complement** of an angle is a right angle minus the given angle. Thus (Fig. in Art. 44), the complement of $A C D$ is $A C F - A C D = D C F$.

43. The **Supplement** of an angle is two right angles minus the given angle. Thus (Fig. Art. 44), the supplement of $A C D$ is $(A C F + F C B) \dot{-} A C D = D C B$.

THEOREM II.

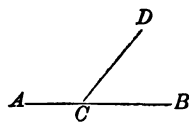
44. *The sum of all the angles formed at a point on one side of a straight line, in the same plane, is equal to two right angles.*



Let DC and EC meet the straight line AB at the point C; then $ACD + DCE + ECB =$ two right angles.

At C erect the perpendicular, CF; then it is evident that

$$ACD + DCE + ECB = ACD + DCF + FCE + ECB$$
$$= ACF + FCB = \text{two right angles.}$$



45. *Cor.* 1. If only two angles are formed, each is the supplement of the other.

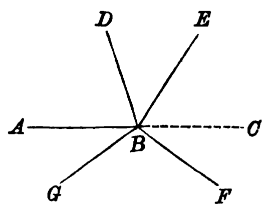
For by the theorem,

$ACD + DCB =$ two angles;

therefore $ACD =$ two right angles $- DCB$,

or $DCB =$ two right angles $- ACD$.

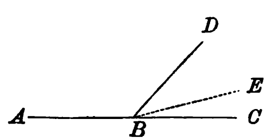
46. *Cor.* 2. The sum of all the angles formed in a plane about a point is equal to four right angles.



Let the angles ABD, DBE, EBF, FBG, GBA, be formed in the same plane about the point B. Produce AB; then the sum of the angles above the line AC is equal to two right angles; and also, the sum of the angles below the line AC is equal to two right angles (44); therefore the sum of all the angles at the point B is equal to four right angles.

THEOREM III.

47. *If at a point in a straight line two other straight lines upon opposite sides of it make the sum of the adjacent angles equal to two right angles, these two lines form a straight line.*



Let the straight line DB meet the two lines, AB, BC, so as to make $ABD + DBC =$ two right angles: then AB and BC form a straight line.

For if $A\,B$ and $B\,C$ do not form a straight line, draw $B\,E$ so that $A\,B$ and $B\,E$ shall form a straight line; then

$$A\,B\,D + D\,B\,E = \text{two right angles (44)};$$

but by hypothesis,

$$A\,B\,D + D\,B\,C = \text{two right angles};$$

therefore $$D\,B\,E = D\,B\,C$$

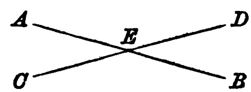
the part equal to the whole, which is absurd (26); therefore $A\,B$ and $B\,C$ form a straight line.

THEOREM IV.

48. *If two straight lines cut each other, the opposite, or vertical, angles are equal.*

Let the two lines, $A\,B$, $C\,D$, cut each other at E; then

$$A\,E\,C = D\,E\,B.$$



For $A\,E\,D$ is the supplement of both $A\,E\,C$ and $D\,E\,B$ (45); therefore

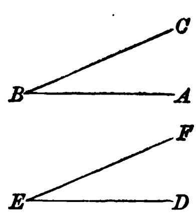
$$A\,E\,C = D\,E\,B$$

In the same way it may be proved that

$$A\,E\,D = C\,E\,B$$

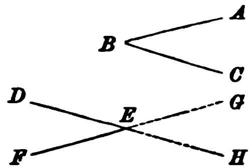
THEOREM V.

49. *Two angles whose sides have the same or opposite directions are equal.*



1st. Let $B\,A$ and $B\,C$, including the angle B, have respectively the same direction as $E\,D$ and $E\,F$, including the angle E; then angle $B =$ angle E.

For since $B\,A$ has the same direction as $E\,D$, and $B\,C$ the same as $E\,F$, the difference of direction of $B\,A$ and $B\,C$ must be the same as the difference of direction of $E\,D$ and $E\,F$; that is, angle $B =$ angle E.



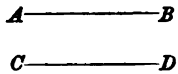
2d. Let BA and BC, including the angle B, have respectively opposite directions to EF and ED, including the angle E; then angle $B =$ angle E.

Produce DE and FE so as to form the angle GEH; then (48)

$$GEH = DEF$$

and $GEH = ABC$ by the first part of this proposition; therefore angle $B =$ angle E.

PARALLEL LINES.



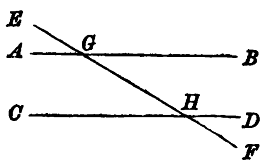
50. *Definition.* **Parallel Lines** are such as have the same direction; as AB and CD.

51. *Corollary.* Parallel lines can never meet. For, since parallel lines have the same direction, if they coincided at one point, they would coincide throughout and form one and the same straight line.

Conversely, straight lines in the same plane that never meet, however far produced, are parallel. For if they never meet they cannot be approaching in either direction, that is, they must have the same direction.

52. *Axiom.* Two lines parallel to a third are parallel to each other.

53. *Definition.* When parallel lines are cut by a third, the angles without the parallels are called *external;* those within, *internal;* thus, AGE, EGB, CHF, FHD are *external* angles; AGH, BGH, GHC, GHD are *internal* angles. Two internal angles on the same side of the secant, or cutting line, are called *internal angles on the same side;* as AGH and GHC, or BGH and GHD. Two internal angles on opposite sides of the secant, and not adjacent, are called *alternate internal angles;* as AGH and GHD, or BGH and GHC.



Two angles, one external, one internal, on the same side of the secant, and not adjacent, are called *opposite external and internal angles;* as EGA and GHC, or EGB and GHD.

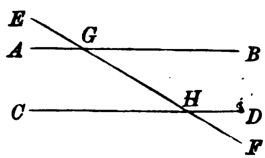
THEOREM VI.

54. *If a straight line cut two parallel lines,*

1st. *The opposite external and internal angles are equal.*

2d. *The alternate internal angles are equal.*

3d. *The internal angles on the same side are supplements of each other.*



Let EF cut the two parallels AB and CD; then

1st. The opposite external and internal angles, EGA and GHC, or EGB and GHD, are equal, since their sides have respectively the same directions (49).

2d. The alternate internal angles, AGH and GHD, or BGH and GHC, are equal, since their sides have opposite directions (49).

3d. The internal angles on the same side, AGH and GHC, or BGH and GHD, are supplements of each other; for AGH is the supplement of AGE (45), which has just been proved equal to GHC. In the same way it may be proved that BGH and GHD are supplements of each other.

55. *Cor.* If a straight line cut two parallel lines, the four acute angles formed are equal; and also the four obtuse angles; and of these angles any acute angle and obtuse angle are supplements of each other.

THEOREM VII.

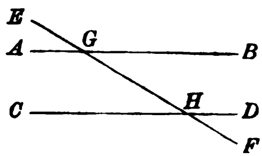
CONVERSE OF THEOREM VI.

56. *If a straight line cut two other straight lines in the same plane, these two lines are parallel,*

1st. *If the opposite external and internal angles are equal.*

2d. *If the alternate internal angles are equal.*

3d. *If the internal angles on the same side are supplements of each other.*



Let EF cut the two lines AB and CD so as to make $EGB = GHD$, or $AGH = GHD$, or BGH and GHD supplements of each other; then AB is parallel to CD.

For, if through the point G a line is drawn parallel to CD, it will make the opposite external and internal angles equal, and the alternate internal angles equal, and the internal angles on the same side supplements of each other (54); therefore it must coincide with AB; that is, AB is parallel to CD.

THEOREM VIII.

57. *But one perpendicular can be drawn from a point to a straight line.*

For, whether the point be within or without the line, if there could be two perpendiculars, they would be parallel to each other (56); and as they coincide at one point, their origin, they must coincide throughout, and form one and the same straight line.

58. *Scholium.* When the point is in the line, the statement must be limited by the words, *in the same plane.* (See Corollary to Theorem I.)

PLANE FIGURES.

DEFINITIONS.

59. A **Plane Figure** is a portion of a plane bounded by lines either straight or curved.

When the bounding lines are straight, the figure is a *polygon*, and the sum of the bounding lines is the *perimeter*.

60. An **Equilateral Polygon** is one whose sides are equal.

61. An **Equiangular Polygon** is one whose angles are equal.

62. Polygons whose sides are respectively equal are *mutually equilateral.*

63. Polygons whose angles are respectively equal are *mutually equiangular.*

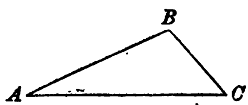
Two equal sides, or two equal angles, one in each polygon, similarly situated, are called *homologous* sides, or angles.

64. **Equal Polygons** are those which, being applied to each other, exactly coincide.

65. Of Polygons, the simplest has three sides, and is called a *triangle;* one of four sides is called a *quadrilateral;* one of five, a *pentagon;* one of six, a *hexagon;* one of eight, an *octagon;* one of ten, a *decagon.*

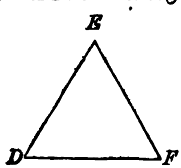
TRIANGLES.

66. A **Scalene Triangle** is one which has no two of its sides equal; as $A B C$.

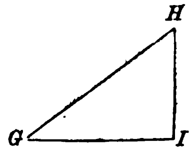


67. An **Isosceles Triangle** is one which has two of its sides equal; as $D E F$.

68. An **Equilateral Triangle** is one whose sides are all equal; as $D E F$.

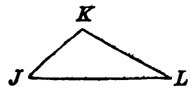


69. A **Right Triangle** is one which has a right angle; as $G\,H\,I$.



The side opposite the right angle is called the *hypothenuse.*

70. An **Obtuse-angled Triangle** is one which has an obtuse angle; as $J\,K\,L$.



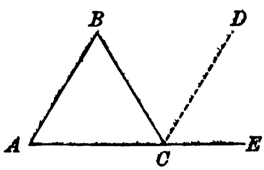
71. An **Acute-angled Triangle** is one whose angles are all acute; as $D\,E\,F$.

Acute and obtuse-angled triangles are called *oblique-angled triangles.*

72. The side upon which any polygon is supposed to stand is generally called its *base;* but in an isosceles triangle, as $D\,E\,F$, in which $D\,E = E\,F$, the third side $D\,F$ is always considered the base.

THEOREM IX.

73. *The sum of the angles of a triangle is equal to two right angles.*



Let $A\,B\,C$ be a triangle; the sum of its three angles, A, B, C, is equal to two right angles.

Produce $A\,C$, and draw $C\,D$ parallel to $A\,B$; then $D\,C\,E = A$, being opposite external and internal angles (54); $B\,C\,D = B$, being alternate internal angles (54);

hence $\quad D\,C\,E + B\,C\,D + B\,C\,A = A + B + B\,C\,A$

but $\quad D\,C\,E + B\,C\,D + B\,C\,A =$ two right angles (44);

therefore $\quad A + B + B\,C\,A =$ two right angles.

74. *Cor.* 1. If two angles of a triangle are known, the third can be found by subtracting their sum from two right angles.

75. *Cor.* 2. If two triangles have two angles of the one respectively equal to two angles of the other, the remaining angles are equal.

76. *Cor.* 3. In a triangle there can be but one right angle, or one obtuse angle.

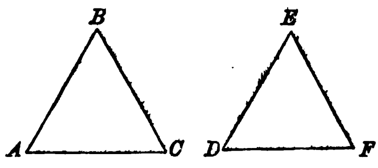
77. *Cor.* 4. In a right triangle the sum of the two acute angles is equal to a right angle.

78. *Cor.* 5. Each angle of an equiangular triangle is equal to one third of two right angles, or two thirds of one right angle.

79. *Cor.* 6. If any side of a triangle is produced, the exterior angle, as $B\,C\,E$, is equal to the sum of the two interior and opposite; therefore it is greater than either of them.

THEOREM X.

80. *If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, the two triangles are equal in all respects.*

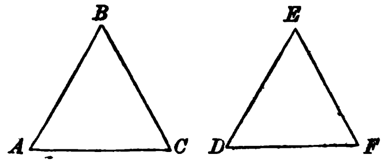


In the triangles $A\,B\,C$, $D\,E\,F$, let the side $A\,B$ equal $D\,E$, $A\,C$ equal $D\,F$, and the angle A equal the angle D; then the triangle $A\,B\,C$ is equal in all respects to the triangle $D\,E\,F$.

Place the side $A\,B$ on $D\,E$, with the point A on the point D, the point B will be on the point E, as $A\,B$ is equal to $D\,E$; then, as the angle A is equal to the angle D, $A\,C$ will take the direction $D\,F$, and as $A\,C$ is equal to $D\,F$, the point C will be on the point F; and $B\,C$ will coincide with $E\,F$. Therefore the two triangles coincide, and are equal in all respects.

THEOREM XI.

81. *If two triangles have two angles and a side of the one respectively equal to two angles and the homologous side of the other, the two triangles are equal in all respects.*



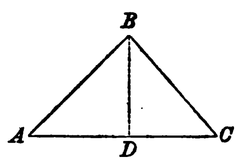
In the triangles $A\,B\,C$ and $D\,E\,F$, let the angle A equal the angle D, the angle B equal the angle E, and the side $A\,C$ equal $D\,F$; then the triangle $A\,B\,C$ is equal in all respects to the triangle $D\,E\,F$.

For, first, since the angles A and B are respectively equal to D and E, the angle C is equal to F (75).

Now place the side $A\,C$ on $D\,F$, with the point A on the point D, the point C will be on the point F, as $A\,C$ is equal to $D\,F$; then, as the angle A is equal to the angle D, $A\,B$ will take the direction $D\,E$, and as the angle C is equal to the angle F, $C\,B$ will take the direction $F\,E$; and the point B falling at once in each of the lines $D\,E$ and $F\,E$ must be at their point of intersection E. Therefore the two triangles coincide, and are equal in all respects.

THEOREM XII.

82. *In an isosceles triangle the angles opposite the equal sides are equal.*



In the isosceles triangle $A\,B\,C$ let $A\,B$ and $B\,C$ be the equal sides; then the angle A is equal to the angle C.

Bisect the angle $A\,B\,C$ by the line $B\,D$; then the triangles $A\,B\,D$ and $B\,C\,D$ are equal, since they have the two sides $A\,B$, $B\,D$, and the included angle $A\,B\,D$ equal respectively to $B\,C$, $B\,D$, and the included angle $D\,B\,C$ (80); therefore the angle $A = C$.

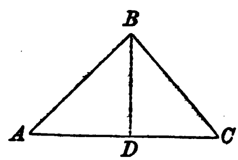
83. *Cor.* 1. From the equality of the triangles $A\,B\,D$ and $B\,C\,D$, $A\,D = D\,C$, and the angle $A\,D\,B = B\,D\,C$. In an isosceles triangle, therefore, a line fulfilling any one of the following conditions fulfils them all:

1. Bisecting the vertical angle.
2. Bisecting the base at right angles.
3. Drawn from the vertex bisecting the base.
4. Drawn from the vertex perpendicular to the base.
5. Drawn from the vertex bisecting the triangle.
6. Bisecting the triangle and the base.
7. Bisecting the triangle and perpendicular to the base.

84. *Cor.* 2. An equilateral triangle is equiangular.

THEOREM XIII.

85. *If two angles of a triangle are equal, the sides opposite are also equal.*



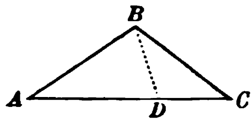
In the triangle $A\,B\,C$ let the angle A equal the angle C; then $A\,B$ is equal to $B\,C$.

Bisect the angle $A\,B\,C$ by the line $B\,D$. Now by hypothesis the angle A is equal to the angle C, and by construction the angle $A\,B\,D$ is equal to the angle $D\,B\,C$; therefore the two triangles $A\,B\,D$, $D\,B\,C$, having two angles and a side, $B\,D$, of the one respectively equal to two angles and the homologous side, $B\,D$, of the other, are equal (81); therefore $A\,B = B\,C$.

86. *Cor.* An equiangular triangle is equilateral.

THEOREM XIV.

87. *The greater side of a triangle is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.*



In the triangle $A B C$ let B be greater than C; then the side $A C$ is greater than $A B$.

At the point B make the angle $C B D$ equal to the angle C;

then (85) $$D B = D C$$

and $$A C = A D + D C = A D + D B$$

But (29)

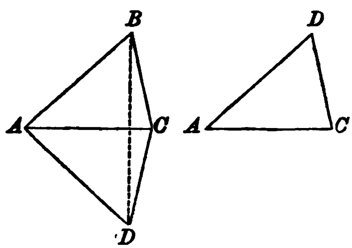
$$A D + D B > A B$$

therefore $$A C > A B$$

Conversely. Let $A C > A B$; then the angle $A B C > C$. Cut off $A D = A B$ and join $B D$; then as $A D = A B$, the angle $A B D = A D B$ (82); and $A D B > C$ (79); therefore $A B D > C$; but $A B C > A B D$; therefore $A B C > C$.

THEOREM XV.

88. *Two triangles mutually equilateral are equal in all respects.*



Let the triangle $A B C$ have $A B$, $B C$, $C A$ respectively equal to $A D$, $D C$, $C A$ of the triangle $A D C$; then $A B C$ is equal in all respects to $A D C$.

Place the triangle $A D C$ so that the base $A C$ will coincide with its equal $A C$, but so that the vertex D will be on the side of $A C$, opposite to B, and the side $C D$ adjacent to its equal $C B$. Join $B D$. Since by hypothesis $A B = A D$, $A B D$ is an isosceles triangle; and the angle $A B D = A D B$ (82); also, since $B C = C D$, $B C D$ is an isosceles triangle; and the angle $D B C = C D B$; therefore the whole angle $A B C = A D C$; therefore the triangles $A B C$ and $A D C$, having two sides and the included angle of the one

equal to two sides and the included angle of the other, are equal (80).

89. *Scholium.* In equal triangles the equal angles are opposite the equal sides.

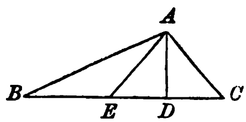
THEOREM XVI.

90. *If from a point without a straight line a perpendicular and oblique lines be drawn to this line,*

1st. *The perpendicular is shorter than any oblique line.*

2d. *Any two oblique lines cutting off equal distances from the foot of the perpendicular are equal.*

3d. *Of two oblique lines the more remote is the greater.*



Let A be the given point, $B C$ the given line, $A D$ the perpendicular, and $A E$, $A B$, $A C$ oblique lines.

1st. In the triangle $A D E$, the angle $A D E$ being a right angle is greater than the angle $A E D$; therefore $A D < A E$ (87).

2d. If $D E = D C$; then the two triangles $A D E$ and $A D C$, having two sides $A D$, $D E$, and the included angle $A D E$ respectively equal to the two sides $A D$, $D C$, and the included angle $A D C$, are equal (80), and $A E$ is equal to $A C$.

3d. If $D B > D E$; then, since $A E D$ is an acute angle, $A E B$ is obtuse, and must therefore be greater than $A B E$ (76); hence $A B > A E$ (87).

91. *Cor.* 1. (Converse of 1st part.) The shortest line from a point to a line is the perpendicular.

92. *Cor.* 2. (Converse of 2d part.) Two equal oblique lines cut off equal distances from the foot of the perpendicular.

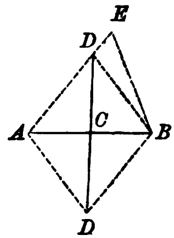
93. *Definition.* The distance from a point to a line means the shortest distance. Therefore the perpendicular measures *the distance* from a point to a line.

THEOREM XVII.

94. *If at the middle of a straight line a perpendicular is drawn,*

1st. *Any point in the perpendicular is equally distant from the extremities of the line.*

2d. *Any point without the perpendicular is unequally distant from the same extremities.*



Let $C D$ be the perpendicular at the middle of the line $A B$; then

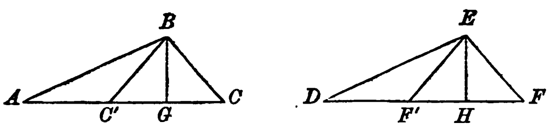
1st. Let D be any point in the perpendicular; draw $D A$ and $D B$. Since $C A = C B$, $D A = D B$ (90).

2d. Let E be any point without the perpendicular; draw $E A$ and $E B$, and from the point D, where $E A$ cuts $D C$, draw $D B$. The angle $A B E > A B D = B A D$; hence, in the triangle $A E B$, since the angle $A B E > B A E$, $E A > E B$ (87).

95. *Cor.* A point not in the perpendicular which bisects a line is on the same side of the perpendicular as the extremity to which it is nearer.

THEOREM XVIII.

96. *If a triangle has two sides and an angle opposite one of these sides respectively equal to two sides and the homologous angle of another triangle, these two triangles are equal in all respects, provided the homologous angles opposite the other given sides are both acute or both obtuse.*



In the triangles $A B C$, $D E F$, let the side $A B$ equal $D E$, $B C$ equal $E F$, and the angle A equal the angle D; then, if

the angles C and F are both acute, or both obtuse, the triangle $A B C$ is equal in all respects to $D E F$.

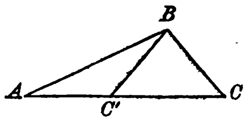
Draw $B G$ and $E H$ perpendicular respectively to $A C$ and $D F$. Now if $B C$ and $E F$ are on the side of the perpendiculars opposite $A B$ and $D E$, the angles C and F are both acute; for as the angles $B G C$ and $E H F$ are both right angles, C and F are both acute angles (76). If $B C$ and $E F$ are on the same side of the perpendiculars as $A B$ and $D E$, that is, in the position $B C'$ and $E F'$, the angles $B C'A$ and $E F'D$ are both obtuse; for as the angles $B G C'$ and $E H F'$ are both right angles, $B C'G$ and $E F'H$ are both acute (76); therefore $B C'A$ and $E F'D$ must both be obtuse (45). Therefore $B C$ is on the same side of the perpendicular as $E F$ if the angles opposite $A B$ and DE are both acute or both obtuse.

Place the side $A B$ on $D E$ with A on D; as $A B$ is equal to $D E$, B will be on E; and as the angle A is equal to D, $A C$ will fall on $D F$; therefore the perpendicular $B G$ will coincide $E H$ (57). Now as $B C$ is equal to $E F$, the points C and F are equally distant from the perpendiculars $B G$ and $E H$ (92); therefore when $B C$ and $E F$ are on the same side of the perpendiculars, that is, when the angles C and F are both acute or both obtuse, $B C$ will fall on $E F$, and the triangles $A B C$, $D E F$ will coincide and be equal in all respects.

97. *Cor.* 1. If in each triangle the side opposite the given angle is greater than the other given side, the angle opposite the latter must be acute (87, 76); therefore *two triangles having two sides and an angle opposite the greater of these two sides equal respectively to two sides and the homologous angle of the other are equal in all respects.*

98. *Cor.* 2. Since if the given angle is not acute, the side opposite must be the greater (76, 87), therefore *two right-angled triangles having the hypothenuse and a side of the one respectively equal to the hypothenuse and a side of the other are equal in all respects.*

99. *Cor.* 3. Two triangles having two sides and an angle opposite the less of these two sides respectively equal are not necessarily equal. For example, the triangles $A\,B\,C$ and $A\,B\,C'$, having $A\,B$, $B\,C$, and the angle A respectively equal to $A\,B$, $B\,C'$, and the angle A of the other, are not equal; but the angle C and $B\,C'A$ are supplements of each other.

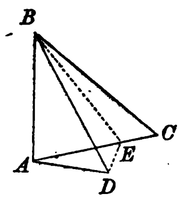


100. *Cor.* 4. From (80), (81), (88), and (96), it follows that with the exception of the ambiguity mentioned in (99), two triangles are equal if any three parts of the one, of which one part is a side, are equal to the corresponding parts of the other.

THEOREM XIX.

101. *If two triangles have two sides of one respectively equal to two sides of the other, but the included angles unequal, the third side of the one having the included angle greater is greater than the third side of the other.*

Let $A\,B$, $B\,C$, two sides of the triangle $A\,B\,C$, be respectively equal to $A\,B$, $B\,D$, two sides of the triangle $A\,B\,D$, but the angle $A\,B\,C > A\,B\,D$; then $A\,C > A\,D$.



Place the triangles with their equal sides $A\,B$ in coincidence, and $B\,C$, $B\,D$, on the same side of $A\,B$; draw $B\,E$ bisecting the angle $D\,B\,C$, and join $D\,E$. The triangles $D\,B\,E$, $E\,B\,C$, are equal (80);

hence $$D\,E = E\,C$$

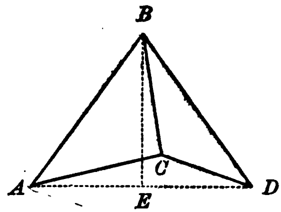
But (29) $$A\,E + E\,D > A\,D$$

Hence $$A\,E + E\,C = A\,C > A\,D$$

THEOREM XX.

CONVERSE OF THEOREM XIX.

102. *If two triangles have two sides of one respectively equal to two sides of the other, but the third sides unequal, the angle opposite this third side is greater in the triangle in which the third side is greater.*

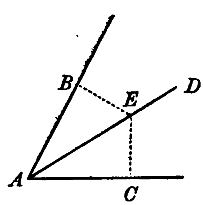


Let $A B$, $B C$, two sides of the triangle $A B C$, be respectively equal to $B D$, $B C$ of the triangle $B D C$, but $A C > C D$; then the angle $A B C > C B D$.

Place the triangles with their equal sides $B C$ in coincidence and $A B$, $B D$ on opposite sides of $B C$; join $A D$ and draw $B E$ perpendicular to $A D$. As $A B$ equals $B D$, $A B D$ is an isosceles triangle, and angle $A B E = E B D$ (83); but as $A C > C D$, C is on the side of the perpendicular nearer $B D$ (95); hence, angle $A B C > A B E$ and $E B D > C B D$ (26); much more then is $A B C > C B D$.

THEOREM XXI.

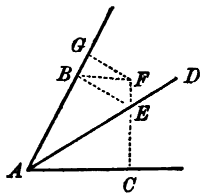
103. *Every point in a line bisecting an angle is equally distant from the sides of the angle; and every point without the bisecting line but within the angle is unequally distant from the sides of the angle.*



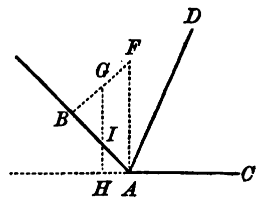
1st. Let E be any point in the line $A D$ which bisects the angle $B A C$; E is equally distant from $A B$ and $A C$.

Draw $E B$, $E C$ perpendicular respectively to $A B$, $A C$. The triangles $A B E$, $A E C$, being right angled at B and C and having the acute angles $B A E$, $E A C$ equal, and $A E$ common, are equal (81); hence $E B = E C$.

2d. Let F be a point out of the bisecting line $A D$, but within the angle; then F is unequally distant from $A B$ and $A C$.



Draw $F G$, $F C$ perpendicular respectively to $A B$, $A C$. If $B A C$ is an acute angle, one of the perpendiculars will cut the bisecting line. Let $F C$ cut $A D$ in E. Draw $E B$ perpendicular to $A B$ and join $B F$. By the first part of the proposition $E B = E C$; hence, $F E + E B = F C$. But (29) $F E + E B > F B$, and (90) $F B > F G$; hence, $F C > F G$. If $B A C$ is an obtuse angle, the point F may be so situated that the perpendicular from F to $A C$ may fall at A, in which case (90) $F A > F B$. Or the given point may be so situated, as at G, that the perpendicular from the point G to $A C$ may fall on $C A$ produced, as $G H$; then we have



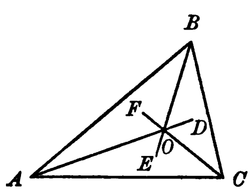
$$G H > G I > G B$$

104. *Corollary.* Conversely, every point equally distant from the sides of an angle is in the line bisecting the angle.

THEOREM XXII.

105. *The three lines bisecting the angles of a triangle intersect at the same point within the triangle.*

Let $A D$, $B E$, $C F$, bisect the angles A, B, C, respectively; $A D$, $B E$, $C F$, intersect at the same point O, within the triangle $A B C$.



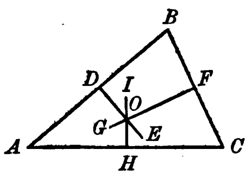
As the angle $D A B < C A B$ and $A B E < A B C$, $A D$ and $B E$ must

meet within the triangle $A\,B\,C$. Let them intersect at O; then O being in $A\,D$ is equally distant from $A\,B$ and $A\,C$ (103); and being in $B\,E$, it is equally distant from $B\,A$ and $B\,C$; therefore O is equally distant from $C\,B$ and $C\,A$ and must (104) be in the line $C\,F$ which bisects the angle C; that is, the lines bisecting the angles of the triangle $A\,B\,C$ intersect at the same point O.

106. *Corollary.* The point O is equally distant from all the sides of the triangle $A\,B\,C$.

THEOREM XXIII.

107. ***The three perpendiculars bisecting the sides of a triangle intersect at the same point.***



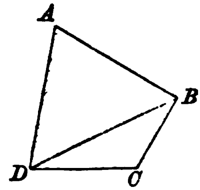
Let $D\,E$, $F\,G$, $H\,I$, be the perpendiculars bisecting respectively $A\,B$, $B\,C$, $C\,A$; then $D\,E$, $F\,G$, $H\,I$, intersect at the same point.

$D\,E$, $H\,I$, must intersect at some point; otherwise they would be parallel (51) and $A\,B$, $A\,C$ would form one and the same straight line, which is impossible, as they are two of the sides of a triangle. Let $D\,E$, $H\,I$ intersect at O. As O is in the perpendicular $D\,E$ which bisects $A\,B$, it is equally distant from A and B (94); and as O is in the perpendicular $H\,I$, which bisects $A\,C$, it is equally distant from A and C; therefore O being equally distant from B and C must be in $F\,G$, the perpendicular which bisects $B\,C$ (94); that is, the perpendiculars bisecting the sides of the triangle $A\,B\,C$ intersect at the same point O.

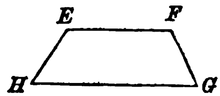
108. *Corollary.* The point O is equally distant from all the vertices of the triangle $A\,B\,C$.

QUADRILATERALS.

DEFINITIONS.

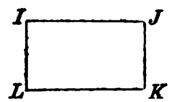


109. A **Trapezium** is a quadrilateral which has no two of its sides parallel; as $A\,B\,C\,D$.

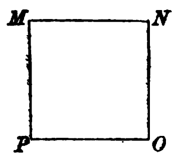


110. A **Trapezoid** is a quadrilateral which has only two of its sides parallel; as $E\,F\,G\,H$.

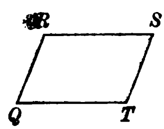
111. A **Parallelogram** is a quadrilateral whose opposite sides are parallel; as $I\,J\,K\,L$, or $M\,N\,O\,P$, or $Q\,R\,S\,T$, or $U\,V\,W\,X$.



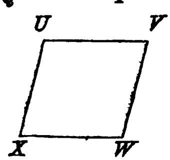
112. A **Rectangle** is a right-angled parallelogram; as $I\,J\,K\,L$.



113. A **Square** is an equilateral rectangle; as $M\,N\,O\,P$.



114. A **Rhomboid** is an oblique-angled parallelogram; as $Q\,R\,S\,T$.

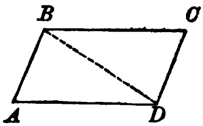


115. A **Rhombus** is an equilateral rhomboid; as $U\,V\,W\,X$.

116. A **Diagonal** is a line joining the vertices of two angles not adjacent; as $D\,B$.

THEOREM XXIV.

117. *In a parallelogram the opposite sides are equal, and the opposite angles are equal.*



Let $ABCD$ be a parallelogram; then will $AB = DC$, $BC = AD$, the angle $A = C$, and $B = D$.

Draw the diagonal BD. As BC and AD are parallel, the alternate angles CBD and BDA are equal (54); and as AB and DC are parallel, the alternate angles ABD and BDC are equal; therefore the two triangles ABD and BDC, having the two angles equal, and the included side BD common, are equal (81); and the sides opposite the equal angles are equal, viz.: $AB = DC$ and $BC = AD$; also the angle $A = C$, and the angle

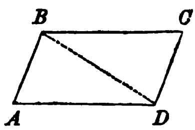
$$ABC = ABD + DBC = BDC + BDA = ADC$$

118. *Cor.* 1. The diagonal divides a parallelogram into two equal triangles.

119. *Cor.* 2. Parallels included between parallels are equal.

THEOREM XXV.

120. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

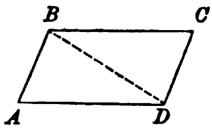


Let $ABCD$ be a quadrilateral having BC equal and parallel to AD; then $ABCD$ is a parallelogram.

Draw the diagonal BD. As BC is parallel to AD, the alternate angles CBD and BDA are equal (54); therefore the two triangles CBD and BDA, having the two sides CB, BD, and the included angle CBD respectively equal to the two sides AD, DB, and the included angle ADB, are equal (80), and the alternate angles ABD and BDC are equal; therefore AB is parallel to DC (56), and $ABCD$ is a parallelogram.

THEOREM XXVI.

121. *If the opposite sides of a quadrilateral are equal the figure is a parallelogram.*

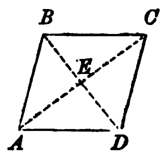


Let $A\,B\,C\,D$ be a quadrilateral having $A\,B = D\,C$ and $B\,C = A\,D$; then $A\,B\,C\,D$ is a parallelogram.

Draw the diagonal $B\,D$. As $A\,B = D\,C$ and $B\,C = A\,D$ and $B\,D$ is common, the two triangles $A\,B\,D$ and $B\,C\,D$ are equal (88); therefore (89) the angle $A\,B\,D = B\,D\,C$, and $A\,B$ is parallel to $D\,C$ (56); and the angle $A\,D\,B = D\,B\,C$, and $A\,D$ is parallel to $B\,C$; therefore $A\,B\,C\,D$ is a parallelogram.

THEOREM XXVII.

122. *The diagonals of a parallelogram bisect each other.*



Let $A\,C$, $B\,D$, be the diagonals of the parallelogram $A\,B\,C\,D$; $A\,C$, $B\,D$, bisect each other at E.

For the triangles $B\,C\,E$, $A\,E\,D$, having $B\,C$ and its adjacent angles (117, 54) equal respectively to $A\,D$ and its adjacent angles, are equal (81);

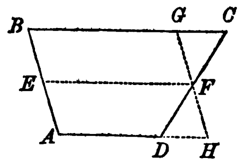
hence (89), $\qquad B\,E = E\,D$, and $A\,E = E\,C$.

123. *Corollary.* The diagonals of a rhombus bisect each other at right angles. For if $A\,B\,C\,D$ is a rhombus, $A\,B$ and $B\,C$ are equal, and $A\,B\,C$ is an isosceles triangle, and $B\,E$ a line from the vertex bisecting the base $A\,C$; therefore $B\,E$ is perpendicular to $A\,C$ (83).

THEOREM XXVIII.

124. *The line joining the middle points of the two sides of a trapezoid which are not parallel is parallel to the two parallel sides, and equal to half their sum.*

Let EF join the middle points of the sides AB and CD, which are not parallel, of the trapezoid $ABCD$; then



1st. EF is parallel to BC and AD. Through F draw GH parallel to BA, meeting AD produced in H. The angles GFC and DFH are equal (48); also the angles GCF and FDH (54); and the side CF is equal to FD; therefore the triangles GFC and DFH are equal (81), and

$$GF = FH = \tfrac{1}{2}GH$$

But as $ABGH$ is a parallelogram, $GH = BA$ (117); therefore

$$FH = \tfrac{1}{2}BA = AE$$

therefore $AEFH$ is a parallelogram (121), and EF is parallel to AD, and therefore also to BC.

2d. $$EF = \tfrac{1}{2}(AD + BC)$$

For as $AEFH$ and $EBGF$ are parallelograms

$$EF = AH = AD + DH$$

and also $$EF = BG = BC - GC$$

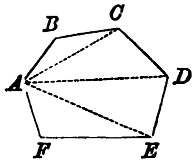
Now, as the two triangles GFC and DFH are equal, $GC = DH$; therefore, if we add the two equations, we shall have

$$2\,EF = AD + BC$$

or $$EF = \tfrac{1}{2}(AD + BC)$$

THEOREM XXIX.

125. *The sum of the interior angles of a polygon is equal to twice as many right angles as it has sides minus two.*



Let $A B C D E F$ be the given polygon; the sum of all the interior angles A, B, C, D, E, F, is equal to twice as many right angles as the figure has sides minus two.

For if from any vertex A, diagonals $A C$, $A D$, $A E$, are drawn, the polygon will be divided into as many triangles as it has sides minus two; and the sum of the angles of each triangle is equal to two right angles (73); therefore the sum of the angles of all the triangles, that is, the sum of the interior angles of the polygon, is equal to twice as many right angles as the polygon has sides minus two.

DEFINITIONS.

126. Every proposition has an *hypothesis* (19), and a *conclusion.* Thus in Proposition XII., *In an isosceles triangle the angles opposite the equal sides are equal,* the hypothesis is that *the triangle is isosceles,* that is, *has two of its sides equal,* and the conclusion is that *the angles opposite the equal sides are equal.*

127. The *converse* of a proposition is a second proposition whose hypothesis and conclusion are respectively the conclusion and hypothesis of the first. Thus, the converse of Proposition XII. is Proposition XIII., in which the hypothesis is that *the two angles of the triangle are equal,* and the conclusion that *the sides opposite are equal,* that is, that *the triangle is isosceles.*

The converse of a true proposition is not necessarily true. For example, the converse of Proposition XXII. is, that *three*

lines drawn from the same point to the vertices of a triangle bisect its angles, which is not a true proposition.

128. Every proposition should be first stated in general terms; then, if demonstrated by the aid of a diagram, should follow the particular statement referring to the representative diagram, and then the drawing of whatever lines are necessary for the demonstration. The drawing of these lines is called *the construction.*

In the process of demonstration by the pupil, it is better not to draw lines of construction till needed for the proof.

129. A *direct demonstration* proceeds from the premises by regular deduction from principles already established.

An *indirect demonstration,* or *reductio ad absurdum,* is one in which the proposition is proved by showing that with the given hypothesis every statement at variance with the given conclusion leads to a result contrary to principles already established.

Though the direct method is generally preferred, the indirect is equally valid. In this Book the indirect method has been used in Theorem III. In this proposition it is evident that AB and BC either *do,* or *do not,* form a straight line. On the supposition that they do not, it follows that a part is equal to the whole, which is contrary to a principle already established (26). It follows therefore that AB and BC do form a straight line.

In some cases there may be several statements at variance with the given conclusion. If, however, these are such that one *must* be true, and only one *can* be true, then when all but one are proved false, this one must be true.

130. *Mathematical Reasoning* may be either analytic or synthetic.

Analysis is the process of reasoning usually adopted in the

discovery of the solution of a problem, or of the proof of a theorem. In analysis the conclusion of a proposition is assumed as true and the consequences are traced until they terminate in the hypothesis of the proposition or in truths already established.

Synthesis is the reverse of analysis; it begins where analysis ends. In synthesis a proposition is proved by building up from truths already established.

In solving the problems of algebra the process is analytic. In the demonstration of propositions as given in elementary geometry the reasoning is synthetic.

Most of the Exercises given on pages 32 – 34 are so simple, — each depending upon but one or two principles already made familiar in this Book, — that there will be no need of the analytic process.

PRACTICAL QUESTIONS.

1. Do two lines that do not meet form an angle with each other? Two lines not in the same plane?

2. Does the magnitude of an angle depend upon the length of its sides?

3. If a right angle is 90°, what is the complement of an angle of 27°? of 51°? of 91°? of 153°? What is the supplement of an angle of 13°? of 83°? of 97°? of 217°?

4. If three of four angles formed at a point on the same side of a straight line, in the same plane, contain respectively 15°, 27°, and 99°, how many degrees does the fourth angle contain?

5. If five of six angles formed in a plane about a point are respectively 11°, 53°, 74°, 19°, and 117°, how many degrees are there in the sixth angle?

6. On opposite sides of a line AB are two lines making with AB, at the point A, the first an angle of 29°, and the second an angle of 61°; how are these two lines related?

7. Can two polygons, each not equilateral, be *mutually* equilateral?

8. Can two polygons, each not equiangular, be *mutually* equiangular?

9. If two angles of a triangle are respectively 32° and 43°, how many degrees are there in the remaining angle?

10. If one acute angle of a right triangle is 24°, how many degrees are there in the other acute angle?

11. How many degrees in each angle of an equiangular triangle?

12. How many degrees in each angle at the base of an isosceles triangle whose vertical angle is 14°?

13. How many degrees in each acute angle of a right-angled isosceles triangle?

14. If one of the angles at the base of an isosceles triangle is double the angle at the vertex, how many degrees in each?

15. If the angle at the vertex of an isosceles triangle is double one of the angles at the base, how many degrees in each?

16. Two triangles mutually equilateral are mutually equiangular (48). Are two triangles mutually equiangular also mutually equilateral?

17. Is a square a parallelogram? Is a parallelogram a square?

18. Is a rectangle a parallelogram? Is a parallelogram a rectangle?

19. How many sides equal to one another can there be in a trapezoid? How many in a trapezium?

20. How many degrees in each angle of an equiangular pentagon? an equiangular hexagon? octagon? decagon? dodecagon?

21. If the parallel sides of a trapezoid are respectively 8 feet and 13 feet in length, how long is the line joining the middle points of the other two sides?

22. If one of the angles of a parallelogram is 120°, how many degrees are there in each of the other angles?

EXERCISES.

The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

131. Two angles whose sides have, one pair the same, the other opposite directions, are supplements of each other. (49; 45.)

132. Any side of a triangle is less than the sum, but greater than the difference, of the other two. (29.)

133. The sum of the lines drawn from a point within a triangle to the extremities of one of the sides is less than the sum of the other two sides. (Produce one of the lines to the side of the triangle.) (29.)

134. Two straight lines perpendicular to a third are parallel. (56.)

135. The sum of the lines drawn from any point within a triangle to the vertices is less than the sum, but greater than half the sum of the sides. (132; 133.)

136. The angle included by the lines drawn from a point within a triangle to the extremities of one of the sides is greater than the angle included by the other two sides.

Produce as in (135). (79.)

137. The angle at the base of an isosceles triangle being one fourth of the angle at the vertex, if a perpendicular is drawn to the base at its extreme point meeting the opposite side produced, the triangle formed by the perpendicular, the side produced, and the remaining side of the triangle is equilateral.

138. If an isosceles and an equilateral triangle have the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer and the extremities of the base, then the angle at the base of the isosceles triangle is $\frac{1}{4}$ or $\frac{5}{2}$ of its vertical angle, according as it is the inner or the outer triangle.

139. Prove Theorem IX. by first drawing a line through B parallel to $A\,C$.

Why is not this method of proof adopted in (73)?

140. Prove Theorem IX. by drawing a triangle upon the floor, walking over its perimeter, and turning at each vertex through an angle equal to the angle at that vertex.

141. If a line joining two parallels is bisected, any other line drawn through the point of bisection and joining the parallels is bisected.

142. If on each side of a triangle an equiangular triangle is constructed externally to the triangle, the straight lines drawn from the remote vertices of the equilateral triangles to the opposite angles of the given triangle are equal.

143. The three perpendiculars from the vertices of a triangle to the opposite sides intersect at the same point.

Through the vertices draw lines parallel to the opposite sides forming a second triangle. (117 ; 107.)

144. A line from the vertex of the right angle of a right triangle bisecting the hypothenuse is equal to half of the hypothenuse.

Let $A\,B\,C$ be the triangle right-angled at C, and from C adjacent to the side $B\,C$ cut off an angle equal to B. (77 ; 85.)

145. If one of the acute angles of a right triangle is double the other, the hypothenuse is double the shortest side. (144; 82; 86.)

146. Prove in Theorem XV. the angles of the two triangles equal by reference to (102); then that the triangles are equal by (80) or (81).

147. (Converse of part of 117.) If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

148. (Converse of 118.) If a diagonal divides a quadrilateral into two equal triangles, is the figure necessarily a parallelogram?

149. (Converse of 122.) If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

150. (Converse of 123.) If the diagonals of a quadrilateral bisect each other at right angles, the figure is a rhombus or a square.

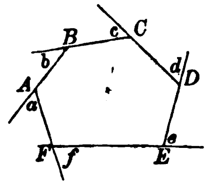
151. The diagonals of a rectangle are equal.

152. The diagonals of a rhombus bisect the angles of the rhombus.

153. Straight lines bisecting the adjacent angles of a parallelogram are perpendicular to each other.

154. From the vertices of a parallelogram measure equal distances upon the sides in order. The lines joining these points on the sides form a parallelogram.

155. Prove Theorem XXIX. by joining any point within to the vertices of the polygon.



156. If the sides of a polygon, as $ABCDEF$, are produced, the sum of the angles a, b, c, d, e, f, is equal to four right angles.

Prove by reference to (125) and (44). Also by drawing from any point lines parallel to the several sides forming the exterior angles, and referring to (46) and (49).

157. If a pavement is to be laid with blocks of the same regular form, that is, blocks whose faces are equiangular and equilateral, prove that their upper faces must be equilateral triangles, squares, or hexagons. (125 ; 46.)

158. If two kinds of regular figures, with sides of the same length, are to be used at each angular point, show that the pavement can be laid only with blocks whose upper faces are,

1st. Triangles and squares.
2d. Triangles and hexagons.
3d. Triangles and dodecagons.
4th. Squares and octagons.

How many of each must there be at each angular point?

159. If three kinds of regular figures, with sides of the same length, are to be used at each angular point, show that the pavement can be laid only with blocks whose upper faces are,

1st. Triangles, squares, and hexagons.
2d. Squares, hexagons, and dodecagons.

How many of each must there be at each angular point?

BOOK II.

RELATIONS OF POLYGONS.

RATIO AND PROPORTION.

[As it is necessary to understand the elementary principles of ratio and proportion before entering upon the Books that are to follow, it is introduced here, though it belongs properly to Algebra.]

DEFINITIONS.

1. Ratio is the relation of one quantity to another of the same kind; or it is the quotient which arises from dividing one quantity by another of the same kind.

Ratio is indicated by writing the two quantities after one another with two dots between, or by expressing the division in the form of a fraction. Thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$; read, a is to b, or a divided by b.

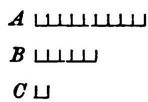
2. The **Terms** of a ratio are the quantities compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, the other the *consequent*; the two terms together are called a *couplet*.

3. An **Inverse** or **Reciprocal Ratio** of any two quantities is the ratio of their *reciprocals.* Thus, the *direct* ratio of a to b is $a : b$, that is, $\frac{a}{b}$; the *inverse* ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, that is, $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b : a$.

4. Two quantities are *commensurable* if there is a third quantity of the same kind which is contained an exact number of times in each. This third quantity is called the *common measure* of these two quantities.

Thus, the two lines A and B are commensurable if there is a third line C which is contained an exact number of times in each, as for example, 9 times in A, and 5 times in B; and the third line C is the common measure of A and B. The ratio of two commensurable quantities therefore can always be exactly expressed in numbers. The ratio of A to B is $\frac{9}{5}$.



Two quantities are *incommensurable* if they have no common measure.

The ratio of two quantities, as A and B, whether commensurable or not, is expressed by $\frac{A}{B}$. If A and B are incommensurable, $\frac{A}{B}$ is called an *incommensurable ratio.*

A *constant ratio* is a ratio which remains the same though its terms may vary. Thus, the ratio of $3 : 4$, $6 : 8$, $9 : 12$, is constant; also the ratio of $A : B$ and $mA : mB$.

5. **Proportion** is an equality of ratios. Four quantities are in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=), or by four dots (: :).

Thus, $a : b = c : d$, or $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$; read a to b equals c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

In a proportion the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proportion. The first and fourth terms are called the *extremes*, and the second and third the *means*.

6. When three quantities are in proportion, e. g. $a : b = b : c$, the second is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

7. A proportion is transformed by **Alternation** when antecedent is compared with antecedent, and consequent with consequent.

8. A proportion is transformed by **Inversion** when the antecedents are made consequents, and the consequents antecedents.

9. A proportion is transformed by **Composition** when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

10. A proportion is transformed by **Division** when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

11. *Axiom.* Two ratios respectively equal to a third are equal to each other.

THEOREM I.

12. *In a proportion the product of the extremes is equal to the product of the means.*

Let	$a : b = c : d$
that is	$\frac{a}{b} = \frac{c}{d}$
Clearing of fractions	$ad = bc$

13. *Scholium.* A proportion is an equation; and making the product of the extremes equal to the product of the means is merely clearing the equation of fractions.

THEOREM II.

14. *If the product of two quantities is equal to the product of two others, the factors of either product may be made the extremes, and the factors of the other the means of a proportion.*

Let	$ad = bc$
Dividing by bd	$\frac{a}{b} = \frac{c}{d}$
that is	$a : b = c : d$

THEOREM III.

15. *If four quantities are in proportion, they will be in proportion by alternation.*

Let	$a : b = c : d$
By (12)	$ad = bc$
By (14)	$a : c = b : d$

THEOREM IV.

16. *If four quantities are in proportion, they will be in proportion by inversion.*

Let $a : b = c : d$

By (12) $ad = bc$

By (14) $b : a = d : c$

THEOREM V.

17. *If four quantities are in proportion, they will be in proportion by composition.*

Let $a : b = c : d$

that is $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member $\frac{a}{b} + 1 = \frac{c}{d} + 1$

or $\frac{a+b}{b} = \frac{c+d}{d}$

that is $a + b : b = c + d : d$

THEOREM VI.

18. *If four quantities are in proportion, they will be in proportion by division.*

Let $a : b = c : d$

that is $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each member $\frac{a}{b} - 1 = \frac{c}{d} - 1$

or $\frac{a-b}{b} = \frac{c-d}{d}$

that is $a - b : b = c - d : d$

19. *Corollary.* From (17) and (18), by means of (15) and (11),

If $$a : b = c : d$$
then $$a + b : a - b = c + d : c - d$$

THEOREM VII.

20. *Equimultiples of two quantities have the same ratio as the quantities themselves.*

For $$\frac{a}{b} = \frac{m\,a}{m\,b}$$
that is $$a : b = m\,a : m\,b$$

21. *Corollary.* It follows that either couplet of a proportion may be multiplied or divided by any quantity, and the resulting quantities will be in proportion. And since by (15), if $a : b = m\,a : m\,b$, $a : m\,a = b : m\,b$ or $m\,a : a = m\,b : b$, it follows that both consequents, or both antecedents, may be multiplied or divided by any quantity, and the resulting quantities will be in proportion.

THEOREM VIII.

22. *If four quantities are in proportion, like powers or like roots of these quantities will be in proportion.*

Let $$a : b = c : d$$
that is $$\frac{a}{b} = \frac{c}{d}$$
Hence $$\frac{a^n}{b^n} = \frac{c^n}{d^n}$$
that is $$a^n : b^n = c^n : d^n$$

Since n may be either integral or fractional, the theorem is proved.

THEOREM IX.

23. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let	$a : b = c : d = e : f$	
Now	$ab = ab$	(A)
and by (12)	$ad = bc$	(B)
and also	$af = be$	(C)
Adding (A), (B), (C)	$a(b + d + f) = b(a + c + e)$	
Hence, by (14)	$a : b = a + c + e : b + d + f$	

THEOREM X.

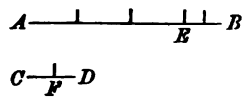
24. *If there are two sets of quantities in proportion, their products, or quotients, term by term, will be in proportion.*

Let	$a : b = c : d$	
and	$e : f = g : h$	
By (12)	$ad = bc$	(A)
and	$eh = fg$	(B)
Multiplying (A) by (B)	$adeh = bcfg$	(C)
Dividing (A) by (B)	$\frac{ad}{eh} = \frac{bc}{fg}$	(D)
From (C) by (14)	$ae : bf = cg : dh$	
and from (D)	$\frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}$	

PROPOSITION XI.

PROBLEM.

25. *To find the numerical ratio of two straight lines.*



Let AB and CD be two straight lines whose numerical ratio is required. From the greater AB cut off as many parts as possible equal to the less CD; for example, three, with a remainder, EB. From CD cut off as many parts as possible equal to EB; for example, one, with a remainder FD. From EB cut off as many parts as possible equal to FD.

If there is still a remainder, continue this process until a remainder is found which is exactly contained in the preceding remainder. This last remainder is a common measure of the two given lines. Suppose FD is contained in EB twice without any remainder; then FD is a common measure of AB and CD; and we have

$$EB = 2\,FD$$

$$CD = EB + FD = 2\,FD + FD = 3\,FD$$

$$AB = 3\,CD + EB = 9\,FD + 2\,FD = 11\,FD.$$

The numerical ratio of AB to $CD = 11\,FD$ to $3\,FD = 11$ to 3.

26. *Corollary.* The common measure thus found is the *greatest* common measure. For the greatest common measure of two lines, AB and CD, cannot be greater than the less, CD; and any common measure of AB and CD must also be a common measure of CD and EB; for it must be contained an exact number of times in CD, and also in AE which is a multiple of CD, and hence, to be a measure of AB, it must also be a measure of EB. Therefore the *greatest* common measure of AB and CD must be the *greatest* common measure of CD and EB. In like manner it can be shown that the

greatest common measure of $C\,D$ and $E\,B$ must be the greatest common measure of $E\,B$ and $F\,D$; and so on.

27. *Scholium.* It may be that, however far this process is continued, no remainder can be found which is exactly contained in the preceding remainder. In this case the quantities are incommensurable. But although the ratio of two incommensurable quantities cannot be exactly expressed in numbers, it may be expressed approximately to any required degree of accuracy.

Suppose, for example, it is required to find the ratio of two incommensurable quantities, A and B, to a degree of accuracy within $\frac{1}{100}$. Let the less, B, be divided into 100 equal parts, and suppose A contains 217 such parts with a remainder less than one of the parts; then we have

$$\frac{A}{B}=\frac{217}{100} \text{ within } \frac{1}{100},$$

that is, $\frac{217}{100}$ is the approximate ratio of A to B to the required degree of accuracy.

Or, to make the reasoning general, let B be divided into n equal parts, and suppose A contains m such parts with a remainder less than one of the parts; then we have

$$\frac{A}{B}=\frac{m}{n} \text{ within } \frac{1}{n}.$$

As n may be taken as great as we please, $\frac{1}{n}$ may be made as small as we please, and $\frac{m}{n}$ will be the ratio of A to B to any required degree of accuracy.

THEOREM XII.

28. *Two incommensurable ratios are equal, if their approximate numerical values are always equal when both ratios are expressed to the same degree of accuracy.*

Let $\frac{A}{B}$ and $\frac{C}{D}$ be two incommensurable ratios whose approximate numerical values are always the same when expressed to the same degree of accuracy; then

$$\frac{A}{B}=\frac{C}{D}$$

Let the numerical ratio $\frac{m}{n}=\frac{A}{B}$, accurate within $\frac{1}{n}$;

then by hypothesis $\frac{m}{n}=\frac{C}{D}$, accurate within $\frac{1}{n}$.

That is, $\frac{A}{B}$ and $\frac{C}{D}$ differ by a quantity less than $\frac{1}{n}$. But $\frac{1}{n}$ may be assumed as small as we please, that is, less than any assignable quantity however small; hence $\frac{A}{B}$ and $\frac{C}{D}$ cannot differ by any assignable quantity however small; that is, $\frac{A}{B}=\frac{C}{D}$.

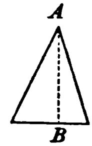
RELATIONS OF POLYGONS.

DEFINITIONS.

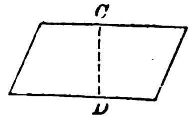
29. The **Area** of a polygon is the measure of its surface. It is expressed in units, which represent the number of times the polygon contains the square unit that is taken as a standard.

30. **Equivalent Polygons** are those which have the same area.

The algebraic sign $=$ is used for either equal or equivalent.

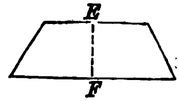


31. The **Altitude** of a triangle is the perpendicular distance from the opposite vertex to the base, or to the base produced; as $A\ B$.



32. The **Altitude** of a parallelogram is the perpendicular distance from the opposite side to the base; as CD.

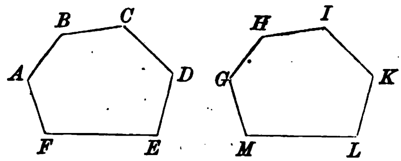
33. The **Altitude** of a trapezoid is the perpendicular distance between its parallel sides; as EF.



THEOREM XIII.

34. *Two polygons mutually equiangular and equilateral are equal.*

Let $ABCDEF$ and $GHIKLM$ be two polygons having the sides AB, BC, CD, DE, EF, FA and the angles A, B, C, D, E, F of the one respectively equal to the sides GH, HI, IK, KL, LM, MG, and the angles G, H, I, K, L, M of the other; then is the polygon $ABCDEF$ equal to the polygon $GHIKLM$.



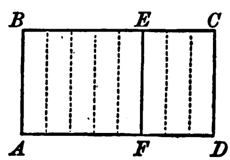
For if the polygon $ABCDEF$ is applied to the polygon $GHIKLM$ so that AB shall be on GH with the point A on G, B will fall on H, as AB and GH are equal; and as the angle B is equal to the angle H, BC will take the direction HI; and as BC is equal to HI, the point C will fall on I; and so also the points D, E, F will fall on the points K, L, M; and the polygon $ABCDEF$ will coincide with the polygon $GHIKLM$, and therefore be equal to it.

THEOREM XIV.

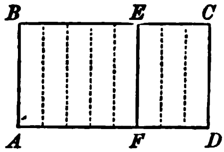
35. *Two rectangles having equal altitudes are to each other as their bases.*

Let $ABCD$, $ABEF$ be two rectangles having equal altitudes, with AD and AF as their bases; then

$$ABCD : ABEF = AD : AF.$$



1st. When the bases have a common measure, which is contained, for example, 8 times in AD and 5 times in AF,



then $AD : AF = 8 : 5$

If, now, AD is divided into 8 equal parts, AF will contain 5 of these parts; and if at the several points of division perpendiculars to the bases are erected, the rectangle $ABCD$ will be divided into 8 equal (34) rectangles, of which $ABEF$ will contain 5; therefore we have

$$ABCD : ABEF = 8 : 5$$

Hence $ABCD : ABEF = AD : AF$

2d. When the bases are incommensurable, suppose the base AD to be divided into m equal parts; then the base AF will contain a certain number n of these parts with a remainder less than one of the parts. Therefore (27)

$$\frac{AD}{AF} = \frac{m}{n} \text{ within } \frac{1}{n}$$

is the numerical expression of the ratio.

If, at the several points of division of the bases, perpendiculars are drawn, the rectangle $ABCD$ will be divided into m equal (34) rectangles, and $ABEF$ into n such rectangles with a remainder less than one of these equal rectangles.

Therefore (27) $\dfrac{ABCD}{ABEF} = \dfrac{m}{n} \text{ within } \dfrac{1}{n}$

Therefore (28) $\dfrac{ABCD}{ABEF} = \dfrac{AD}{AF}$

or, $ABCD : ABEF = AD : AF$

Therefore, whether the bases are commensurable or not, *two rectangles having equal altitudes are as their bases.*

36. *Corollary.* As $A B$ may be considered the base, and $A D$ and $A F$ the altitudes, it follows that *rectangles having equal bases are as their altitudes.*

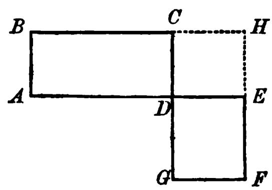
37. *Scholium.* By *rectangle* in these propositions is meant *surface of the rectangle.*

THEOREM XV.

38. *Any two rectangles are to each other as the products of their bases by their altitudes.*

Let $A B C D$, $D E F G$ be two rectangles; then

$$A B C D : D E F G = A D \times D C : D E \times D G$$



Place the two rectangles so that the angles at D are vertical, and produce $B C$, $F E$ till they meet at H; a rectangle $D C H E$ will be thus formed having an altitude equal to the altitude of $A B C D$, and a base equal to the base of $D E F G$. Therefore by (35),

$$A B C D : D C H E = A D : D E$$

and by (36),

$$D C H E : D E F G = D C : D G$$

Hence by (24; 21),

$$A B C D : D E F G = A D \times D C : D E \times D G$$

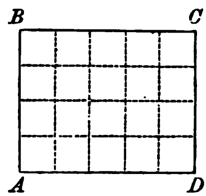
39. *Cor.* 1. Hence, any rectangle is to the square unit taken as a standard (29) as the product of its base by its altitude is to unity; therefore *the area of a rectangle is equal to the product of its base by its altitude, or to the product of its two dimensions.*

40. *Cor.* 2. The area of a square is the square of one of its sides.

41. *Sch.* 1. By *product of the base by the altitude*, or *product of the base and the altitude*, is meant the product of the numbers which express the number of linear units in the base and altitude respectively.

By *square of a side* is meant the second power of the number which expresses the number of linear units in the side.

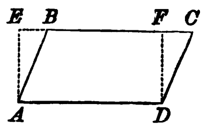
42. *Sch.* 2. When the base and altitude have a common measure, this proposition is made evident by dividing the base and altitude into equal parts representing the linear unit taken. Suppose the base AD contains 5 of these parts and the altitude AB 4; by drawing lines through the points of division parallel to the sides of the rectangle, the rectangle will be divided into squares, each equal to the unit of surface; and the rectangle will evidently contain 5×4, or 20, of these squares; that is, its area $= 5 \times 4 = 20$.



THEOREM XVI.

43. *The area of a parallelogram is equal to the product of its base and altitude.*

Let DF be the altitude of the parallelogram $ABCD$; then the area of $ABCD = AD \times DF$.



At A draw the perpendicular AE meeting CB produced in E; $AEFD$ is a rectangle equivalent to the parallelogram $ABCD$. For the two triangles AEB and DFC, having the sides AE, AB equal respectively to the sides DF, DC (I. 119), and the included angle EAB equal to the included angle FDC (I. 49), are equal. Adding DFC to the common part $ABFD$ gives the parallelogram $ABCD$;

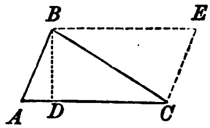
and adding its equal $A\,E\,B$ to the common part $A\,B\,F\,D$, gives the rectangle $A\,E\,F\,D$; therefore the parallelogram $A\,B\,C\,D$ is equivalent to the rectangle $A\,E\,F\,D$; but the area of the rectangle $= A\,D \times D\,F$ (39); therefore the area of the parallelogram $= A\,D \times D\,F$.

44. *Corollary.* Parallelograms having equal bases and equal altitudes are equivalent.

THEOREM XVII.

45. *The area of a triangle is equal to half the product of its base and altitude.*

Let $B\,D$ be the altitude of the triangle $A\,B\,C$; then the area of $A\,B\,C = \frac{1}{2}\,A\,C \times B\,D$.



Draw $C\,E$ parallel to $A\,B$, and $B\,E$ parallel to $A\,C$, forming the parallelogram $A\,B\,E\,C$. The triangle $A\,B\,C$ is one half the parallelogram $A\,B\,E\,C$ (I. 118); the area of the parallelogram $= A\,C \times B\,D$ (43); therefore the area of the triangle $= \frac{1}{2}\,A\,C \times B\,D$.

46. *Cor.* 1 Triangles are to each other as the products of their bases and altitudes. For if A and a represent the altitudes of two triangles T and t, and B and b their bases, their areas are $\frac{1}{2}\,A \times B$ and $\frac{1}{2}\,a \times b$; therefore

$$T : t = \tfrac{1}{2}\,A \times B : \tfrac{1}{2}\,a \times b$$

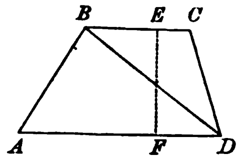
or (21)

$$T : t = A \times B : a \times b$$

47. *Cor.* 2. Triangles having equal bases are as their altitudes; those having equal altitudes as their bases. For in the proportion above, if $B = b$, or $A = a$, the equals can be cancelled from the second ratio (21).

THEOREM XVIII.

48. *The area of a trapezoid is equal to half the product of its altitude by the sum of its parallel sides.*



Let EF be the altitude of the trapezoid $ABCD$; then the area of $ABCD$ $= \frac{1}{2} EF \times (BC + AD)$.

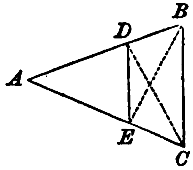
Draw the diagonal BD; it will divide the trapezoid into two triangles, ABD, BCD, having the same altitude EF as the trapezoid.

By (45) the area of $BCD = \frac{1}{2} EF \times BC$
and the area of $ABD = \frac{1}{2} EF \times AD$
Therefore the area of the trapezoid $= \frac{1}{2} EF \times (BC + AD)$.

49. *Corollary.* As (I. 124) the line joining the middle points of the sides AB and CD of the trapezoid $= \frac{1}{2} (BC + AD)$, therefore the area of a trapezoid is equal to the product of its altitude by the line joining the middle points of the sides which are not parallel.

THEOREM XIX.

50. *A line drawn parallel to one side of a triangle divides the other sides proportionally.*



In the triangle ABC let DE be drawn parallel to BC; then

$$AE : EC = AD : DB$$

or $$AC : AE = AB : AD$$

Draw DC and BE; the triangles ADE and EDC, having the same vertex D and their bases in the same straight

line $A\,C$, have the same altitude; therefore (47)

$$A\,D\,E : E\,D\,C = A\,E : E\,C$$

And the triangles $A\,D\,E$ and $D\,E\,B$, having the same vertex E and their bases in the same straight line $A\,B$, have the same altitude; therefore (47)

$$A\,D\,E : D\,E\,B = A\,D : D\,B$$

But the triangles $E\,D\,C$ and $D\,E\,B$ are equivalent (45), since they have the same base $D\,E$, and the same altitude, viz., the perpendicular distance between the two parallels $D\,E$ and $B\,C$.

Therefore (11) $\quad A\,E : E\,C = A\,D : D\,B$

By (17) $\quad A\,E + E\,C : A\,E = A\,D + D\,B : A\,D$

that is $\quad A\,C : A\,E = A\,B : A\,D$

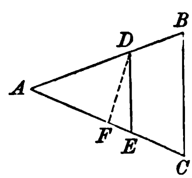
THEOREM XX.

CONVERSE OF THEOREM XIX.

51. *A line dividing two sides of a triangle proportionally is parallel to the third side of the triangle.*

In the triangle $A\,B\,C$ if $D\,E$ divides $A\,B$ and $A\,C$ so that $A\,B : A\,D = A\,C : A\,E$, then $D\,E$ is parallel to $B\,C$.

For if $D\,E$ is not parallel to $B\,C$, through D draw $D\,F$ parallel to $B\,C$; then (50)



$$A\,B : A\,D = A\,C : A\,F$$

But by hypothesis

$$A\,B : A\,D = A\,C : A\,E$$

Now as the first three terms of these two proportions are the same, their fourth terms must be equal; that is, $A\,F = A\,E$, the part equal to the whole, which is absurd: therefore $D\,E$ is parallel to $B\,C$.

DEFINITIONS.

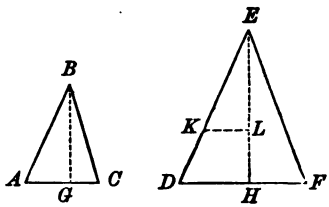
52. Similar Polygons are those whose homologous lines have a constant ratio.

53. Homologous points in similar polygons are points in those polygons similarly situated.

54. Homologous lines in similar polygons are lines in those polygons similarly situated.

THEOREM XXI.

55. *Triangles mutually equiangular are similar.*



In the triangles $A\,B\,C$, $D\,E\,F$, let the angle $A = D$, $B = E$, and $C = F$; then the triangles are similar.

From the homologous vertices B and E draw $B\,G$ and $E\,H$ cutting off like parts of the angles B and E; the angle $A\,B\,G = D\,E\,H$, and $B\,G$ and $E\,H$ are homologous lines.

Cut off $E\,K = B\,A$ and $E\,L = B\,G$, and join $K\,L$; the triangle $K\,E\,L = A\,B\,G$ (I. 80), and the angle $E\,K\,L = A$; but $A = D$; therefore $E\,K\,L = D$, and $K\,L$ is parallel to $D\,H$ (I. 56); and (50)

$$E\,D : E\,K = E\,H : E\,L$$

or

$$E\,D : B\,A = E\,H : B\,G$$

that is, the homologous lines $E\,H$, $B\,G$ have the same ratio as the sides $E\,D$, $B\,A$.

In like manner it can be proved that

$$E\,D : B\,A = E\,F : B\,C = D\,F : A\,C$$

and that any other homologous lines of the triangles $A B C$, $D E F$, have the same ratio to one another as the homologous sides, that is, have a constant ratio; therefore the triangles $A B C$, $D E F$ are similar.

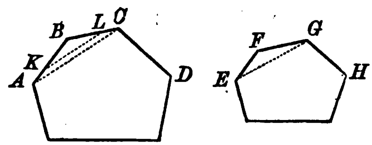
56. *Corollary.* Two triangles whose homologous sides are equally inclined to each other are similar.

For if one of the triangles is turned through an angle equal to the angle of inclination of the sides, the sides of the triangles become respectively parallel; the triangles are therefore mutually equiangular (I. 49), and similar (55).

THEOREM XXII.

57. *Similar polygons are mutually equiangular.*

Let $A D$ and $E H$ be similar polygons; they are mutually equiangular.



Draw the homologous diagonals $A C$, $E G$, cutting off the triangles $A B C$, $E F G$. Cut off $B K = F E$ and $B L = F G$, and join $K L$.

As the polygons are similar (52)

$$B A : F E = B C : F G$$

or $$B A : B K = B C : B L$$

hence $K L$ is parallel to $A C$ (51), and the angle $B K L = B A C$ and $B L K = B C A$ (I. 49); and the triangles $B L K$ and $A B C$ are mutually equiangular, and therefore similar (55).

Now as the polygons are similar

$$B C : F G = A C : E G$$

but as the triangles $A B C$, $K B L$ are similar

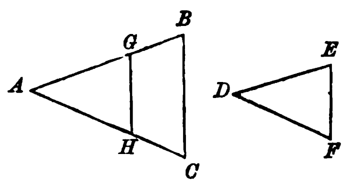
$$B C : B L = A C : K L$$

As the first three terms of these two proportions are the same, or equal, their fourth terms must be equal, that is, $EG = KL$; and the triangle $KBL = EFG$ (I. 88); therefore the angle $B = F$ (I. 89). In like manner it can be proved that the other angles of the polygons are equal respectively; therefore the polygons are mutually equiangular.

58. *Scholium.* This proposition includes the converse of the preceding.

THEOREM XXIII.

59. ***Triangles whose homologous sides have a constant ratio are similar.***



In the triangles ABC, DEF, let

$AB : DE = AC : DF = BC : EF$

then the triangles ABC, DEF, are similar.

From AB cut off $AG = DE$, and draw GH parallel to BC. Then the triangles ABC, AGH, being mutually equiangular, are similar (55), and we have

$$AB : AG = AC : AH$$

But by hypothesis $AB : DE = AC : DF$

As the first three terms of these proportions are the same, or equal, their fourth terms must be equal; that is, $AH = DF$; in like manner it can be proved that $GH = EF$. Therefore the triangle $AGH = DEF$; and as AGH is similar to ABC, its equal, DEF, is also similar to ABC.

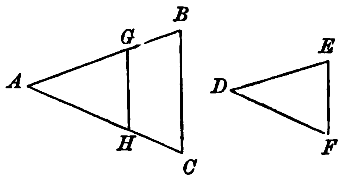
THEOREM XXIV.

60. *Two triangles having an angle of the one equal to an angle of the other, and the sides including these angles proportional, are similar.*

In the triangles ABC, DEF let the angle $A = D$ and

$$AB : DE = AC : DF$$

then the triangles ABC and DEF are similar.



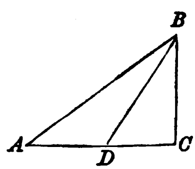
Cut off AG and AH respectively equal to DE and DF, and join GH; the triangle $AGH = DEF$, and the angle $AGH = E$ (I. 80).

By hypothesis $\quad AB : DE = AC : DF$

or $\quad AB : AG = AC : AH$

that is, the sides AB, AC are divided proportionally by the line GH; therefore GH is parallel to BC (51), and the angle $AGH = B$ (I. 54); but the angle $AGH = E$; therefore $B = E$, and the two triangles are mutually equiangular, and therefore similar (55).

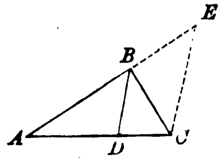
61. *Definition.* When a point is taken on a given line, or on a given line produced, the distances of the point from the extremities of the line are called the *segments* of the line. Thus, in the triangle ABC, the segments of the base AC, made by a line BD cutting the base, are AD and DC; and in the triangle ABD the segments of the base, made by a line BC cutting the base, are AC and DC. If the point is within the given line, the sum of the segments, if in the line produced, the difference of the segments, is equal to the line.



THEOREM XXV.

62. *The line bisecting any angle of a triangle, interior or exterior, divides the opposite side into segments which are proportional to the adjacent sides.*

1st. Let B, an interior angle of the triangle $A\,B\,C$, be bisected by $B\,D$;



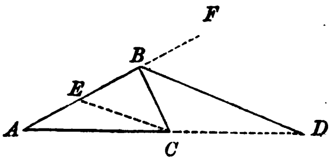
then $\quad A\,B : B\,C = A\,D : D\,C$

Through C draw $C\,E$ parallel to $D\,B$, meeting $A\,B$ produced in E. As $B\,D$ and $E\,C$ are parallel, the angle $A\,B\,D = B\,E\,C$ and $D\,B\,C = B\,C\,E$ (I. 54); but $A\,B\,D = D\,B\,C$; hence $B\,C\,E = B\,E\,C$, and $B\,E = B\,C$. But as $B\,D$ and $E\,C$ are parallel (50)

$$A\,B : B\,E = A\,D : D\,C$$

or, as $B\,E = B\,C$, $\quad A\,B : B\,C = A\,D : D\,C$

2d. Let $C\,B\,F$, an exterior angle of $A\,B\,C$, be bisected by $B\,D$ which meets $A\,C$ produced in D; then



$$A\,B : B\,C = A\,D : D\,C$$

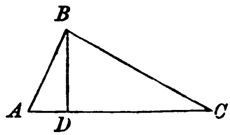
Through C draw $C\,E$ parallel to $D\,B$. As $B\,D$ and $E\,C$ are parallel, the angle $C\,E\,B = D\,B\,F$ and $E\,C\,B = C\,B\,D$ (I. 54); but $C\,B\,D = D\,B\,F$; hence $C\,E\,B = E\,C\,B$, and $B\,E = B\,C$. But as $B\,D$ and $E\,C$ are parallel

$$A\,B : B\,E = A\,D : D\,C$$

or, as $B\,E = B\,C$, $\quad A\,B : B\,C = A\,D : D\,C$

THEOREM XXVI.

63. *In a right triangle the perpendicular drawn from the vertex of the right angle to the hypothenuse divides the triangle into two triangles similar to the whole triangle and to each other.*



In the right triangle $A\,B\,C$, if $B\,D$ is drawn from the vertex B of the right angle perpendicular to the hypothenuse $A\,C$, the two triangles $A\,B\,D$, $B\,C\,D$ are similar to $A\,B\,C$ and to each other.

The two right triangles $A\,B\,D$ and $A\,B\,C$ have the acute angle A common; they are therefore mutually equiangular (I. 75) and similar (55). The two right triangles $A\,B\,C$ and $B\,C\,D$ have the acute angle C common; therefore they are mutually equiangular and similar.

Also the two triangles $A\,B\,D$ and $B\,C\,D$, being each similar to $A\,B\,C$, are similar to each other.

64. *Cor.* 1. Since $A\,B\,C$ and $A\,B\,D$ are similar triangles

$$A\,C : A\,B = A\,B : A\,D$$

And since $A\,B\,C$ and $B\,C\,D$ are similar

$$A\,C : C\,B = C\,B : C\,D$$

that is, *if in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypothenuse, either side about the right angle is a mean proportional between the whole hypothenuse and the adjacent segment.*

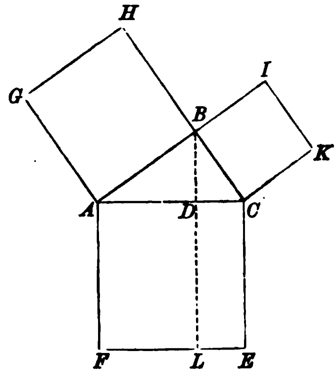
65. *Cor.* 2. As $A\,B\,D$ and $B\,C\,D$ are similar triangles

$$A\,D : D\,B = D\,B : D\,C$$

that is, *in a right triangle the perpendicular from the vertex of the right angle to the hypothenuse is a mean proportional between the segments of the hypothenuse.*

THEOREM XXVII.

66. *The square described on the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the other two sides.*



Let ABC be a triangle right-angled at B; then

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

On the three sides construct squares, draw BD perpendicular to AC, and produce it to FE; $DCEL$ is a rectangle whose area is (39)

$$CE \times CD = AC \times CD$$

The area of the square (40)

$$BIKC = \overline{BC}^2$$

But (64) $\qquad AC : BC = BC : CD$

or $\qquad AC \times CD = \overline{BC}^2$

that is, the square $BIKC$ is equivalent to the rectangle $DCEL$. In the same way the square $AGHB$ can be proved equivalent to the rectangle $ADLF$: therefore the sum of the two rectangles, that is, the square $ACEF$ is equivalent to the sum of the squares $BIKC$ and $AGHB$; or

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

67. *Corollary.* Since

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

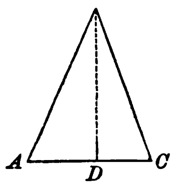
$$\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$$

and $\qquad BC = \sqrt{\overline{AC}^2 - \overline{AB}^2}$

THEOREM XXVIII.

68. *In a triangle the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides minus twice the product of one of these sides and the distance from the vertex of this acute angle to the foot of the perpendicular let fall upon this side from the vertex of the opposite angle.*

Fig. 1.



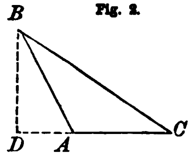
If C is an acute angle of the triangle $A\,B\,C$, and $B\,D$ is the perpendicular from B to $A\,C$, then

$$\overline{A\,B}^2 = \overline{A\,C}^2 + \overline{B\,C}^2 - 2\,A\,C \times D\,C$$

For if $B\,D$ falls within the base (Fig. 1), we have

$$A\,D = A\,C - D\,C$$

Fig. 2.



And if $B\,D$ falls upon the base produced (Fig. 2), we have

$$A\,D = D\,C - A\,C$$

And the square of either of these equations gives

$$\overline{A\,D}^2 = \overline{A\,C}^2 + \overline{D\,C}^2 - 2\,A\,C \times D\,C$$

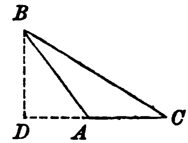
Adding $\overline{B\,D}^2$ to both members and substituting for $\overline{B\,D}^2 + \overline{A\,D}^2$ its equivalent $\overline{A\,B}^2$ (66), and for $\overline{B\,D}^2 + \overline{D\,C}^2$ its equivalent $\overline{B\,C}^2$, we have

$$\overline{A\,B}^2 = \overline{A\,C}^2 + \overline{B\,C}^2 - 2\,A\,C \times D\,C$$

69. *Corollary.* If the angle C becomes a right angle $D\,C = 0$ and $2\,A\,C \times D\,C = 0$, and the proposition reduces to the same as (66).

THEOREM XXIX.

70. *In an obtuse-angled triangle the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the other two sides plus twice the product of one of these sides and the distance from the vertex of the obtuse angle to the foot of the perpendicular let fall upon this side from the vertex of the opposite angle.*



If A is the obtuse angle of the triangle $A\,B\,C$, and $B\,D$ is the perpendicular from B to $A\,C$, then

$$\overline{B\,C}^2 = \overline{A\,C}^2 + \overline{A\,B}^2 + 2\,A\,C \times A\,D$$

In this case the perpendicular $B\,D$ will always fall on the side produced, and we have

$$D\,C = A\,C + A\,D$$

Squaring this we have

$$\overline{D\,C}^2 = \overline{A\,C}^2 + \overline{A\,D}^2 + 2\,A\,C \times A\,D$$

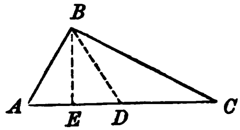
Adding $\overline{B\,D}^2$ to both members, and substituting for $\overline{B\,D}^2 + \overline{D\,C}^2$ its equivalent $\overline{B\,C}^2$, and for $\overline{B\,D}^2 + \overline{A\,D}^2$ its equivalent $\overline{A\,B}^2$, we have

$$\overline{B\,C}^2 = \overline{A\,C}^2 + \overline{A\,B}^2 + 2\,A\,C \times A\,D$$

71. *Corollary.* If the angle A becomes a right angle $A\,D = 0$ and $2\,A\,C \times A\,D = 0$, and the proposition reduces to the same as (66).

THEOREM XXX.

72. *If a line is drawn from the vertex of any angle of a triangle bisecting the opposite side, the sum of the squares of the other two sides is equivalent to twice the square of the bisecting line plus twice the square of a segment of the bisected side.*



In the triangle $A B C$ let $B D$ be drawn bisecting $A C$, and $B E$ perpendicular to $A C$; then

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{BD}^2$$

If $A B < B C$, the angle $B D A$ will be acute and $B D C$ obtuse, and in the triangles $A B D$, $D B C$, by (68) and (70) we have

$$\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2\,A D \times E D$$

$$\overline{BC}^2 = \overline{DC}^2 + \overline{BD}^2 + 2\,D C \times E D$$

Adding these equations, observing that $A D = D C$, we have

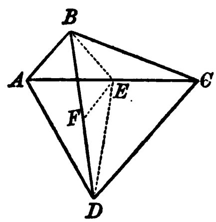
$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{BD}^2$$

THEOREM XXXI.

73. *The sum of the squares of the four sides of a quadrilateral is equivalent to the sum of the squares of its diagonals plus four times the square of the line joining the middle points of the diagonals.*

Let $E F$ join, E, F, the middle points of $A C$, $B D$, the diagonals of the quadrilateral $A B C D$, then

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2$$



Draw $B E$, $E D$. In the triangle $A B C$ (72)

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AE}^2 + 2\,\overline{BE}^2$$

and in the triangle $A C D$

$$\overline{CD}^2 + \overline{DA}^2 = 2\,\overline{AE}^2 + 2\,\overline{DE}^2$$

Adding these equations, we have

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\,\overline{AE}^2 + 2\,\overline{BE}^2 + 2\,\overline{DE}^2$$

But from the triangle $B E D$ (72)

$$2\,\overline{BE}^2 + 2\,\overline{ED}^2 = 4\,\overline{BF}^2 + 4\,\overline{EF}^2$$

Therefore

$$\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2=4\,\overline{AE}^2+4\,\overline{BF}^2+4\,\overline{EF}^2$$

But $4\,\overline{AE}^2=(2\,AE)^2=\overline{AC}^2$, and $4\,\overline{BF}^2=(2\,BF)^2=\overline{BD}^2$
Therefore

$$\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2=\overline{AC}^2+\overline{BD}^2+4\,\overline{EF}^2$$

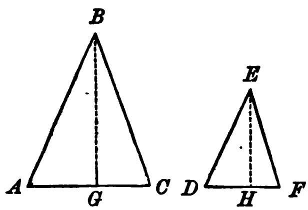
74. *Corollary.* If the quadrilateral $ABCD$ is a parallelogram, the diagonals bisect each other (I. 122); that is, $EF=0$, and the sum of the squares of the sides is equivalent to the sum of the squares of the diagonals.

THEOREM XXXII.

75. *Similar triangles are to each other as the squares of their homologous sides.*

Let ABC and DEF be two similar triangles; then

$$ABC:DEF=\overline{AC}^2:\overline{DF}^2$$



Draw BG and EH perpendicular respectively to AC and DF; then, as BG and EH are homologous lines of the similar triangles ABC, DEF, we have (52)

$$BG:EH=AC:DF$$

this multiplied by the proportion

$$\tfrac{1}{2}AC:\tfrac{1}{2}DF=AC:DF$$

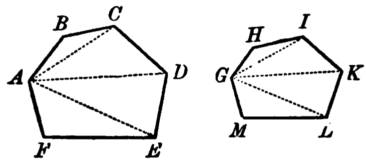
gives $$\tfrac{1}{2}AC\times BG:\tfrac{1}{2}DF\times EH=\overline{AC}^2:\overline{DF}^2$$

but $\frac{1}{2}AC\times BG$ is the area of ABC, and $\frac{1}{2}DF\times EH$ is the area of DEF (45); therefore

$$ABC:DEF=\overline{AC}^2:\overline{DF}^2$$

THEOREM XXXIII.

76. *Polygons that are mutually equiangular and whose homologous sides have a constant ratio are similar.*



Let the angles A, B, C, &c., of the polygon $B\,E$ be equal respectively to G, H, I, &c., of the polygon $H\,L$, and let $A\,B : G\,H = B\,C : H\,I = C\,D : I\,K$, &c.; then $B\,E$ is similar to $H\,L$.

From the homologous vertices A and G draw the diagonals $A\,C$, $A\,D$, $A\,E$, $G\,I$, $G\,K$, and $G\,L$. As by hypothesis the angle $B = H$, and $A\,B : G\,H = B\,C : H\,I$, therefore the triangles $A\,B\,C$ and $G\,H\,I$ are similar (60). As the triangles $A\,B\,C$ and $G\,H\,I$ are similar, the angle $B\,C\,A = H\,I\,G$ (57); but the whole angle $B\,C\,D = H\,I\,K$; therefore the angle $A\,C\,D = G\,I\,K$; and as the triangles $A\,B\,C$ and $G\,H\,I$ are similar

$$B\,C : H\,I = A\,C : G\,I$$

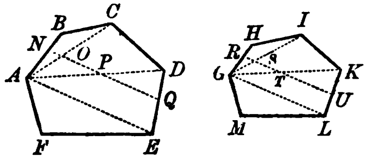
But by hypothesis $\quad B\,C : H\,I = C\,D : I\,K$

Therefore (11) $\quad A\,C : G\,I = C\,D : I\,K$

and $A\,C\,D$ and $G\,I\,K$ are similar (60). In like manner it can be proved that the other triangles are similar each to each; hence all the homologous lines of the two polygons situated in corresponding triangles have a constant ratio.

It only remains to be proved that lines, each not wholly in a single triangle, joining homologous points of the polygons, have this same constant ratio.

From N draw any such line as $N\,Q$, cutting $A\,C$ in O, and $A\,D$ in P; in $G\,H$ and $G\,I$ let R and S be points, homologous respectively to N and O in $A\,B$ and $A\,C$; draw $R\,S$ and produce it cutting $G\,K$ in T and meeting $K\,L$ in U. The triangles



$A\,N\,O$, $G\,R\,S$, being similar (52), are mutually equiangular (57); therefore the exterior angle $A\,O\,P = G\,S\,T$, and the triangles $A\,O\,P$, $G\,S\,T$ are mutually equiangular (I. 75) and similar (55), and $O\,P$ and $S\,T$ homologous lines in the triangles $A\,C\,D$ and $G\,I\,K$. In like manner it can be shown that $P\,Q$ and $T\,U$ are homologous lines in the triangles $A\,D\,E$ and $G\,K\,L$; hence Q and U are homologous points in $D\,E$ and $K\,L$, and $N\,Q$ and $R\,U$ homologous lines of the polygons $B\,E$ and $H\,L$.

Now as $N\,O$ and $R\,S$ are homologous lines of the similar triangles $A\,B\,C$ and $G\,H\,I$

$$N\,O : R\,S = A\,C : G\,I$$

and as $O\,P$ and $S\,T$ are homologous lines of the similar triangles $A\,C\,D$ and $G\,I\,K$

$$O\,P : S\,T = A\,C : G\,I$$

Hence (11) $\qquad N\,O : R\,S = O\,P : S\,T$

In like manner it can be proved that

$$O\,P : S\,T = P\,Q : T\,U$$

Therefore (23)

$$N\,O + O\,P + P\,Q : R\,S + S\,T + T\,U = N\,O : R\,S = A\,C : G\,I$$

or $\qquad N\,Q : R\,U = A\,C : G\,I$

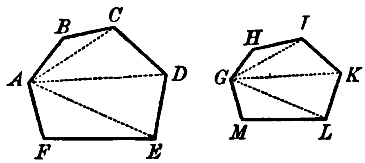
Hence, homologous lines of $B\,E$ and $H\,L$ have a constant ratio, and (52) the polygons are similar.

77. *Cor.* 1. By drawing all the diagonals possible from the vertices of two homologous angles of two similar polygons, these polygons will be divided into the same number of similar triangles.

78. *Cor.* 2. Polygons composed of the same number of similar triangles similarly situated are similar.

THEOREM XXXIV.

79. *The perimeters of similar polygons are to each other as the homologous sides; and the polygons as the squares of the homologous sides.*



Let $A\,B\,C\,D\,E\,F$ and $G\,H\,I\,K\,L\,M$ be two similar polygons.

1st. Their perimeters are to each other as $A\,B : G\,H$.

For as the polygons are similar

$$A\,B : G\,H = B\,C : H\,I = C\,D : I\,K, \text{ \&c.}$$

Therefore (23)

$$A\,B + B\,C + C\,D, \text{ \&c.} : G\,H + H\,I + I\,K, \text{ \&c.} = A\,B : G\,H$$

that is, the perimeters of $A\,B\,C\,D\,E\,F$ and $G\,H\,I\,K\,L\,M$ are as $A\,B : G\,H$.

2d. $\quad A\,B\,C\,D\,E\,F : G\,H\,I\,K\,L\,M = \overline{A\,B}^2 : \overline{G\,H}^2$

From the homologous vertices A and G draw the diagonals $A\,C$, $A\,D$, $A\,E$, $G\,I$, $G\,K$, and $G\,L$; the polygons will be divided into the same number of similar triangles (77); therefore (75)

Triangle	$A\,B\,C : G\,H\,I = \overline{A\,C}^2 : \overline{G\,I}^2$
and	$A\,C\,D : G\,I\,K = \overline{A\,C}^2 : \overline{G\,I}^2$
Therefore	$A\,B\,C : G\,H\,I = A\,C\,D : G\,I\,K$
In like manner	$A\,C\,D : G\,I\,K = A\,D\,E : G\,K\,L$
and	$A\,D\,E : G\,K\,L = A\,E\,F : G\,L\,M$

Hence (23)

$$A\,B\,C + A\,C\,D + A\,D\,E + A\,E\,F : G\,H\,I + G\,I\,K + G\,K\,L + G\,L\,M = A\,B\,C : G\,H\,I$$

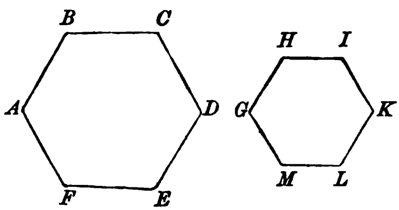
But $\quad A\,B\,C : G\,H\,I = \overline{A\,B}^2 : \overline{G\,H}^2$

Therefore the sums of the triangles, that is, the polygons themselves, are to each other as the squares of the homologous sides.

80. *Definition.* A **Regular Polygon** is one that is both equiangular and equilateral.

THEOREM XXXV.

81. *Regular polygons of the same number of sides are similar.*



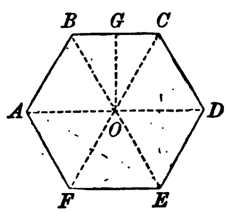
Let $A\,D$ and $G\,K$ be two regular polygons of the same number of sides; they are similar.

They are mutually equiangular; for the sum of their angles is the same (I. 125); and each angle is equal to this sum divided by the number of angles which is the same.

The homologous sides have a constant ratio; for as the polygons are regular, $A\,B = B\,C = C\,D$, &c., and $G\,H = H\,I = I\,K$, &c., therefore $A\,B : G\,H = B\,C : H\,I = C\,D : I\,K$, &c. Therefore $A\,D$ and $G\,F$ are similar (76).

THEOREM XXXVI.

82. *There is a point in a regular polygon equidistant from its vertices, and also equidistant from its sides.*



Let $A\,B\,C\,D\,E\,F$ be a regular polygon. Bisect the angles A and B by $A\,O$ and $B\,O$. As the whole angles A and B are each less than two right angles, the sum of $O\,A\,B$ and $A\,B\,O$ is less than two right angles; therefore $A\,O$ and $B\,O$ cannot be parallel (I. 54), but will meet.

Suppose them to meet in the point O; then O is equidistant from the vertices A, B, C, D, E, F, and also from the sides $A B$, $B C$, $C D$, &c.

Draw $O C$, $O D$, $O E$, $O F$. $O A = O B$ (I. 85). As $O B$ bisects the whole angle B, the angle $O B A = O B C$; therefore the triangle $A B O = O B C$ (I. 80), and $O C = O A = O B$. In like manner it can be proved that $OD = OE = OF = OA$; that is, O is equidistant from the vertices of the polygon.

As the triangles $O A B$, $O B C$, $O C D$, &c., are equal, their altitudes are equal, that is, the bases are equidistant from the vertex O.

83. *Scholium.* O is called *the centre*, $O B$ *the radius*, and the perpendicular $O G$ *the apothem of the polygon.*

84. *Corollary.* In regular polygons of the same number of sides, the apothems are as the homologous sides; therefore *the perimeters of regular polygons of the same number of sides are as their apothems; and the polygons as the squares of their apothems.*

THEOREM XXXVII.

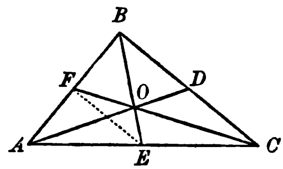
85. *The area of a regular polygon is equal to half the product of its perimeter and apothem.*

For, if a regular polygon is divided into triangles by lines drawn from the centre to the several vertices, the area of each triangle of which the polygon is composed is equal to half the product of its base and the apothem of the polygon (45); therefore the area of the polygon is equal to half the product of the sum of the bases, that is, its perimeter, and its apothem.

THEOREM XXXVIII.

86. *The three lines drawn from the vertices of a triangle bisecting the opposite sides intersect at the same point.*

Let $A D$, $B E$, $C F$, bisect respectively the sides $B C$, $C A$, $A B$, of the triangle $A B C$; then $A D$, $B E$, $C F$, intersect at the same point.



Let BE and CF intersect at O; join FE. As $AB:AF=AC:AE$, FE is parallel to BC (51). Hence the triangles FOE, OBC, are mutually equiangular and similar. Therefore

$$BC:FE=BO:OE$$

But $$BC:FE=AB:AF$$

Hence (11) $$BO:OE=AB:AF$$

But $$AB=2AF$$

Therefore $$BO=2OE$$

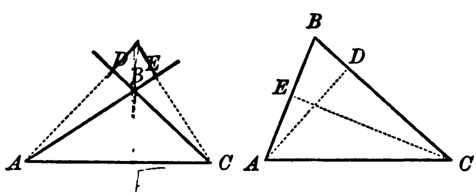
That is, BE and CF intersect at a point O whose distance from B is two thirds of BE. In like manner it can be shown that AD intersects BE at a point whose distance from B is two thirds of BE, that is, at O. Therefore the three lines AD, BE, CF, intersect at the same point.

87. *Scholium.* The point O is the centre of gravity of the triangle ABC.

THEOREM XXXIX.

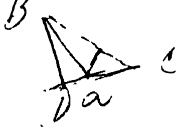
88. *If perpendiculars are drawn from any two vertices of a triangle to the opposite sides (produced, if necessary), the sides are reciprocally proportional to the segments between the vertex of the angle of these sides and the perpendiculars.*

Let AD, CE be perpendiculars from A and C to the sides BC, AB respectively; then



$$AB:BC=BD:BE.$$

For the right triangles ADB, BEC, having an equal (or common) angle at B, are mutually equiangular and (55) similar.

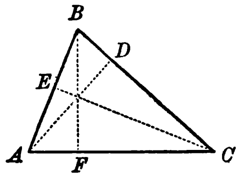


$\overline{BC}^2 = \overline{ac}^2 + \overline{ab}^2 + 2ac \times ad$

Hence, $AB : BC = BD : BE$

89. *Cor.* 1. $AB \times BE = BC \times BD$

90. *Cor.* 2. If AD, BF, CE, are perpendicular respectively to BC, CA, AB, then (89)



$$AC \times AF = AB \times AE$$

and $$AC \times FC = BC \times DC$$

Whence $AC\,(AF + FC) = AB \times AE + BC \times DC$

or $$\overline{AC}^2 = AB \times AE + BC \times DC$$

If B is a right angle, the points E, D, would be at B, and $AE = AB$, $DC = BC$; then this equation becomes

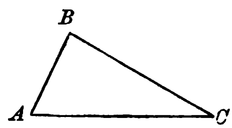
$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

This furnishes another proof of the Pythagorean proposition (66).

THEOREM XL.

91. *Any figure described on the hypothenuse of a right triangle is equivalent to the sum of the similar figures of which the other sides of the triangle are homologous sides.*

Let P represent any figure described on AC, the hypothenuse, and M, N, similar figures whose sides, homologous to the side AC of the figure P, are the sides AB, BC, of the right triangle ABC; then $P = M + N$.



For (79) $$M : N = \overline{AB}^2 : \overline{BC}^2$$

or (17) $$M + N : N = \overline{AB}^2 + \overline{BC}^2 : \overline{BC}^2$$

But (79) $$P : N = \overline{AC}^2 : \overline{BC}^2$$

As $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$ (66), the last three terms of the last two proportions, taken in order, are the same, or equal; hence the first two must be equal, or $P = M + N$.

PRACTICAL QUESTIONS.

1. What is the perimeter and the area of a rectangle 25 by 35 inches?

180 —

2. What is the area of a parallelogram whose base is 20 feet and altitude 12 feet?

3. What is the area of a triangle whose base is 14 feet and altitude 8 feet?

4. What is the square surface of a board 15 feet long, and 16 inches wide at one end and 9 inches at the other? What kind of a figure is it?

5. What integral numbers will express the sides and hypothenuse of a right triangle?

6. How far from a tower 40 feet high must the foot of a ladder 50 feet long be placed that it may exactly reach the top of the tower?

7. The foot of a ladder 67 feet long stands 40 feet from a wall; how much nearer the wall must the foot be placed that the ladder may reach 10 feet higher?

8. If a ladder 108 feet long, with its foot in the street, will reach on one side to a window 75 feet high, and on the other to a window 45 feet high, how wide is the street?

9. A has an acre of land one of whose sides is 20 rods in length; B has a piece of land of exactly similar form containing 9 acres. What is the length of the corresponding side of B's?

10. What is the distance on the floor from one corner to the opposite corner of a rectangular room 16 by 24 feet?

11. If the height of the above room is 10 feet, what is the distance from the lower corner to the opposite upper corner?

12. Find the length of the longest straight rod that can be put into a box whose inner dimensions are 12, 4, and 3.

13. What is the altitude of an equilateral triangle whose side is 12 feet?

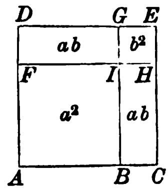
14. If the bases of two similar triangles are respectively 100 and 10 feet, how many triangles equal to the second are equivalent to the first?

15. How many times as much paint will it take to cover a church whose steeple is 120 feet in height as to cover an exact model of the church whose steeple is 10 feet in height?

16. What is the area of a right-angled triangle whose hypothenuse is 125 feet and one of the sides 75 feet?

EXERCISES.

The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

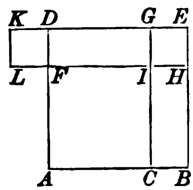


92. The square on the sum $A\,C$ of two straight lines $A\,B$, $B\,C$ is equivalent to the squares on $A\,B$ and $B\,C$, together with twice the rectangle $A\,B.B\,C$.

Or, algebraically, if $a = A\,B$, and $b = B\,C$,

$$(a+b)^2 = a^2 + 2\,a\,b + b^2$$

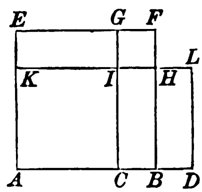
93. *Corollary.* The square on a line is four times the square on half of the line.



94. The square on the difference $A\,C$ of two straight lines $A\,B$, $B\,C$ is equivalent to the squares on $A\,B$ and $B\,C$, diminished by twice the rectangle $A\,B.B\,C$.

Or, algebraically, if $a = A\,B$, and $b = B\,C$,

$$(a-b)^2 = a^2 - 2\,a\,b + b^2$$



95. The rectangle contained by the sum and difference of two lines $A\,B$, $B\,C$ is equivalent to the difference of their squares.

Or, algebraically, if $a = A\,B$ and $b = B\,C$

$$(a+b)\,(a-b) = a^2 - b^2$$

Produce $A\,B$ so that $B\,D = B\,C$.

96. Parallelograms are to each other as the products of their bases and altitudes.

97. Parallelograms having equal bases are to each other as their altitudes; those having equal altitudes are as their bases.

98. Where must a line from the vertex be drawn to bisect a triangle?

99. Two or more lines parallel to the base of a triangle divide the other sides, or the other sides produced, proportionally.

100. Lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram; and the perimeter of this parallelogram is equal to the sum of the diagonals of the quadrilateral.

Draw the diagonals. (51.)

101. Lines drawn from the vertex of a triangle divide the opposite side and a parallel to it proportionally.

102. State and prove the converse of (101).

103. $ABCD$ is a parallelogram; E and F the middle points of AB and CD. BF and ED trisect the diagonal AC.

104. If two triangles have two sides of the one equal respectively to two sides of the other, and the included angles supplementary, the triangles are equivalent.

105. The diagonals divide a parallelogram into four equivalent triangles. Two triangles standing on opposite sides are equal.

106. If the middle points of the sides of a triangle are joined, the area of the triangle thus formed is one fourth the area of the original triangle.

107. Every line passing through the intersection of the diagonals of a parallelogram bisects the parallelogram.

108. If a point within a parallelogram is joined to the vertices, the two triangles formed by the joining lines and two opposite sides are together equivalent to half the parallelogram.

Through the point draw lines parallel to the sides of the parallelogram.

109. State and prove the proposition if the point named in (108) is without the parallelogram.

110. The area of a trapezoid is equal to twice the area of the triangle formed by joining the extremities of one non-parallel side to the middle point of the other.

111. Draw two polygons mutually equiangular but not similar. Draw two polygons not similar, whose sides taken in order have a constant ratio.

112. Two triangles are similar if two angles of the one are equal respectively to two angles of the other.

113. The lines bisecting the angles of a parallelogram form a rectangle whose diagonals are parallel respectively to the sides of the parallelogram.

114. If two triangles have one angle equal, and a second angle supplementary, the sides including their third angles have the same ratio. (Place the equal angles on each other.)

115. If two triangles have one angle equal and the sides about a second angle have the same ratio, their remaining angles are either equal or supplementary.

If in each triangle the side opposite the given equal angle is greater than the side adjacent, or if the given angle is not acute, or if their third angles are both acute, or both obtuse, the triangles are similar. Compare I. 96 – 100.

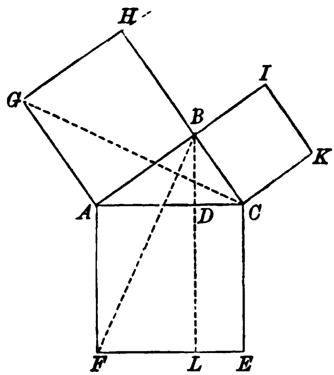
116. Two triangles having an angle of the one equal to an angle in the other are to each other as the rectangles of the sides containing the equal angles; or (Fig. Art. 50)

$$ABC : ADE = AB \times AC : AD \times AE$$

Draw DC. (47 ; 24 ; 21.)

117. Prove Theorem XXVII., first drawing GC and BF; then proving the triangles AGC and ABF equal.

Turn the triangle ABF on the point A in its own plane till AB coincides with AG; where will F be? (39 ; 45.)



118. Prove that if GH, KI, and LB are produced, they will meet in the same point.

119. Prove Theorem XXVII., first producing FA to GH, and producing GH, KI, and LB till they meet.

120. Prove Theorem XXVII., first constructing the squares on opposite sides of $A\,B$ and $B\,C$ from that on which they are drawn in the figure in Art. 117; moving the square $A\,G\,H\,B$ on $A\,B$, a distance equal to $B\,C$ in the direction $B\,A$; then proving that these squares are divided into parts that can be made to coincide with the parts of the square on $A\,C$.

121. In the figure in Art. 117 draw $H\,I$, $K\,E$, $F\,G$. The triangle HIB is equal, and the triangles CKE, GAF are equivalent to ABC.

122. If $G\,F$ and $K\,E$ are drawn, $\overline{G\,F}^2 + \overline{K\,E}^2 = 5\,\overline{A\,C}^2$.

From G and K draw perpendiculars to $F\,A$ and $E\,C$ respectively. (70.)

123. If from any point P straight lines are drawn to the vertices of a rectangle $A\,B\,C\,D$, $\overline{P\,A}^2 + \overline{P\,C}^2 = \overline{P\,B}^2 + \overline{P\,D}^2$.

124. Prove Theorem XXVIII. by means of (90).

125. Prove Theorem XXIX. by means of (90).

126. In the Fig. in Art. 86, if $A\,D$, $B\,E$, $C\,F$ bisect respectively $B\,C$, $C\,A$, $A\,B$,

1st. If $A\,D = C\,F$, $A\,B\,C$ is isosceles.

2d. Any line $G\,H$ drawn from $A\,B$ to $A\,C$ parallel to $B\,C$ is bisected by $A\,D$.

3d. If $G\,C$ and $H\,B$ are drawn they will intersect in $A\,D$.

4th. The triangle constructed with the sides $A\,D$, $B\,E$, $C\,F$, is to $A\,B\,C$ as $3:4$.

5th. $4\,(\overline{A\,D}^2 + \overline{B\,E}^2 + \overline{C\,F}^2) = 3\,(\overline{A\,B}^2 + \overline{B\,C}^2 + \overline{C\,A}^2)$.

127. The squares of the sides of a right triangle are as the segments of the hypothenuse made by a perpendicular from the vertex of the right angle to the hypothenuse.

128. The square of the hypothenuse is to the square of either side as the hypothenuse is to the segment adjacent to this side made by a perpendicular from the vertex of the right angle.

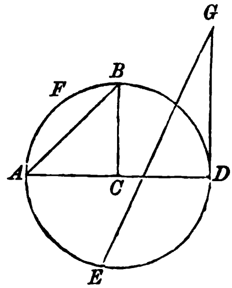
129. The side of a square is to its diagonal as $1:\sqrt{2}$; or the square described on the diagonal of a square is double the square itself.

BOOK III.

THE CIRCLE.

DEFINITIONS.

1. A **Circle** is a plane figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within called the *centre*; as $A\ B\ D\ E$.



2. An **Arc** is any part of the circumference; as $A\ F\ B$.

3. A **Chord** is the straight line joining the ends of an arc; as $A\ B$.

4. The **Diameter** of a circle is a chord passing through the centre; as $A\ D$.

5. The **Radius** of a circle is a line drawn from the centre to the circumference; as $C\ D$.

6. *Corollary.* The radii of a circle, or of equal circles, are equal; also the diameters are equal, and each is equal to double the radius.

7. A **Segment** of a circle is the part of the circle cut off by a chord; as the space included by the arc $A\ F\ B$ and the chord $A\ B$.

8. A **Sector** is the part of a circle included by two radii and the intercepted arc; as the space $B\ C\ D$.

9. A **Tangent** (in geometry) is a line which touches, but does not, though produced, cut the circumference; as $G\ D$.

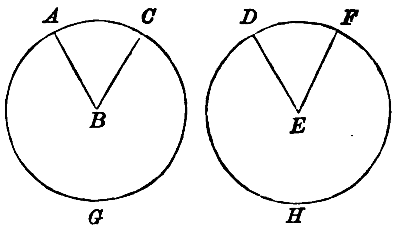
A tangent is often considered as terminating at one end at the point of contact, at the other where it meets another tangent or a secant.

10. A **Secant** (in geometry) is a line lying partly within and partly without a circle; as GE.

A secant is generally considered as terminating at one end where it meets the concave circumference, and at the other where it meets another secant or a tangent.

THEOREM I.

11. *In the same circle, or equal circles, equal angles at the centre are subtended by equal arcs; and, conversely, equal arcs subtend equal angles at the centre.*



Let B and E be equal angles at the centres of the two equal circles ACG and DFH; then the arcs AC and DF are equal.

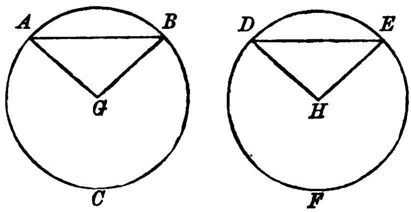
Place the angle B on the angle E; as they are equal they will coincide; and as BA and BC are equal to ED and EF, the point A will coincide with D, and the point C with F; and the arc AC will coincide with DF, otherwise there would be points in the one or the other arc unequally distant from the centre.

Conversely. If the arcs AC and DF are equal, the angles B and E are equal.

For, if the radius AB is placed on the radius DE with the point B on E, the point A will fall on D, as $AB = DE$; and the arc AC will coincide with DF, otherwise there would be points in the one or the other arc unequally distant from the centre; and as the arc $AC = DF$, the point C will fall on F; therefore BC will coincide with EF, and the angle B be equal to E.

THEOREM II.

12. *In the same circle, or equal circles, equal arcs are subtended by equal chords; and conversely, if the chords are equal, the arcs are equal.*



Let ABC and DEF be two equal circles; if the arcs AB and DE are equal, the chords AB and DE are equal.

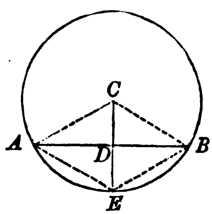
For, if the centre of the circle ABC is placed on the centre of DEF with the point A of the circumference on the point D, as the arcs are equal, B will fall on E, and the chord AB will coincide with DE; therefore $AB = DE$.

Conversely. If the chords AB and DE are equal, the arcs AB and DE are equal.

Draw the radii GA, GB, HD, HE. The triangles ABG, DEH, being mutually equilateral, are equal (I. 88), and the angles at G and H are equal; hence (11) the arcs AB and DE are equal.

THEOREM III.

13. *The radius perpendicular to a chord bisects the chord and the arc subtended by the chord.*



Let CE be a radius perpendicular to the chord AB; it bisects the chord AB, and also the arc AEB.

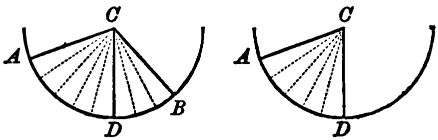
Draw the radii CA and CB and the chords AE and EB. As equal oblique lines are equally distant from the perpendicular, $AD = DB$ (I. 92); and as E is a point in the perpendicular to the middle of AB, it is equally distant from A and B (I. 94); therefore the chords and hence (12) the arcs AE, EB are equal.

14. *Corollary.* In a circle a line fulfilling any one of the following conditions fulfils them all:

1. Drawn through the centre perpendicular to the chord.
2. Drawn through the centre bisecting the chord.
3. Drawn through the centre bisecting the arc.
4. Bisecting the chord perpendicularly.
5. Bisecting the arc and its chord.
6. Bisecting the arc and perpendicular to its chord.

THEOREM IV.

15. *In the same circle, or equal circles, two angles at the centre are as their intercepted arcs.*



Let ACB, ACD be angles at the centre of the same or equal circles; and AB, AD their intercepted arcs; then

$$ACB : ACD = AB : AD$$

1st. When the arcs have a common measure, which is contained, for example, 8 times in AB and 5 times in AD; then

$$AB : AD = 8 : 5$$

For, if AB is divided into 8 equal parts, AD will contain 5 of these parts, and if radii are drawn to the several points of division the angle ACB will be divided into 8 equal angles of which ACD will contain 5; therefore we have

$$ACB : ACD = 8 : 5$$

Hence $$ACB : ACD = AB : AD$$

2d. When the arcs are incommensurable, the proportion is proved by the same method as that used in (II. 35).

16. *Cor.* 1. As angles at the centre vary as their arcs, or arcs as their corresponding angles, either of these quantities is assumed as the measure of the other. The measure of an angle is, then, *the arc included between its sides and described from its vertex as a centre.*

17. *Cor.* 2. As the sum of all the angles about the point C is equal to four right angles (I. 46), one right angle is measured by one quarter of a circumference, or by a quadrant.

18. *Scho.* 1. It should be understood, however, that arcs and angles are unlike quantities, and that the statement in Cor. 1 means that if we adopt any angle as the unit of angles, and its corresponding arc as the unit of arcs, the *numerical* measure of any angle is equal to the *numerical* measure of its corresponding arc; or m times any angle corresponds to m times its arc.

19. *Scho.* 2. The right angle is sometimes adopted as the unit of angle and its corresponding arc, a quadrant, as the unit of arc. But generally the unit of angle is taken as $\frac{1}{90}$ of a right angle, called a degree ($^\circ$); and the corresponding unit of arc $\frac{1}{90}$ of a quadrant, which is also called a degree ($^\circ$). The degree, both of angles and arcs, is subdivided into minutes ($'$), and seconds ($''$), a minute being $\frac{1}{60}$ of a degree, and a second $\frac{1}{60}$ of a minute.

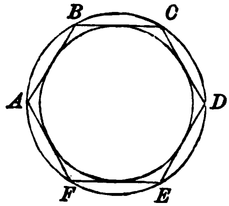
An angle and its corresponding arc are therefore numerically expressed in degrees, minutes, and seconds. Thus, $\frac{1}{4}$ of a right angle, as well as its corresponding arc, will be expressed by $22^\circ\ 30'\ 0''$.

DEFINITIONS.

20. An **Inscribed Angle** is one whose vertex is in the circumference and whose sides are chords; as $A\ B\ C$ in the outer circle.

21. An **Inscribed Polygon** is one whose sides are chords.

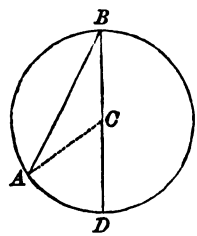
Thus $ABCDEF$ is inscribed in the outer circle. In this case the circle is said to be circumscribed about the polygon.



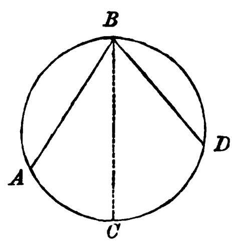
22. A **Circumscribed Polygon** is one whose sides are tangents. Thus $ABCDEF$ is circumscribed about the inner circle. In this case the circle is said to be inscribed in the polygon.

THEOREM V.

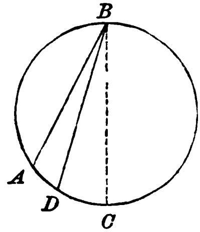
23. *An inscribed angle is measured by half the arc included by its sides.*



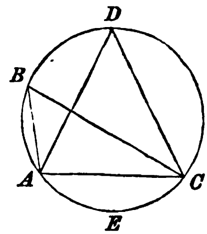
1st. When one of the sides BD is a diameter; then the angle B is measured by half the arc AD. Draw the radius CA, and the triangle ACB is isosceles, CA and CB being radii; therefore the angle $A \doteq B$ (I. 82). But the exterior angle ACD is equal to the sum of the two angles A and B (I. 79); therefore the angle B is equal to half the angle ACD; the angle ACD is measured by the arc AD (16); therefore the angle B is measured by half the arc AD.



2d. When the centre is within the angle, draw the diameter BC. By the preceding part of the proposition the angle ABC is measured by half the arc AC, and CBD by half CD; therefore $ABC + CBD$, or ABD, is measured by half $AC + CD$, or half the arc AD.



3d. When the centre is without the angle, draw the diameter BC. By the first part of the proposition the angle ABC is measured by half the arc AC, and DBC by half DC; therefore $ABC - DBC$, or ABD, is measured by half $AC - DC$, or half the arc AD.

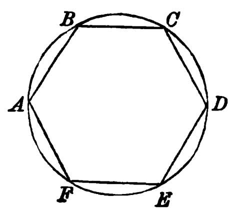


24. *Cor.* 1. All the angles ABC, ADC, inscribed in the same segment are equal; for each is measured by half the arc AEC.

25. *Cor.* 2. Every angle inscribed in a semicircle is a right angle; for it is measured by half a semi-circumference, or by a quadrant (17).

THEOREM VI.

26. *Every equilateral polygon inscribed in a circle is regular.*

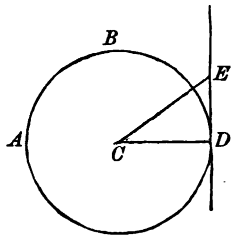


Let $ABCDEF$ be an equilateral polygon inscribed in a circle; it is also equiangular and therefore regular.

For the chords AB, BC, CD, &c. being equal, the arcs AB, BC, CD, &c. are equal (12); therefore the arc $AB +$ the arc BC will be equal to the arc $BC +$ the arc CD, &c.; that is, the angles B, C, &c. are in equal segments; therefore they are equal (24), and the polygon is equiangular and regular.

THEOREM VII.

27. *A tangent to a circumference is perpendicular to the radius drawn to the point of contact.*

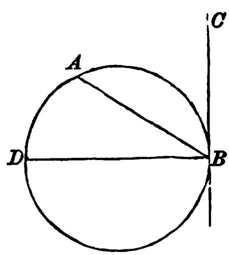


Let $D E$ be a tangent to the circumference $A B D$ at the point D; then $D E$ is perpendicular to the radius $C D$. For if any other line, as $C E$, is drawn to $D E$, as E must be without the circumference, $C E$ will be longer than $C D$; that is, $C D$ is the shortest line from C to $D E$, and is, therefore, perpendicular to $D E$ (I. 91).

28. *Corollary.* Conversely, a straight line perpendicular to a radius at its extremity is a tangent to the circumference (I. 39).

THEOREM VIII.

29. *The angle made by a tangent and a chord, intersecting at the same point of the circumference, is measured by half the included arc.*

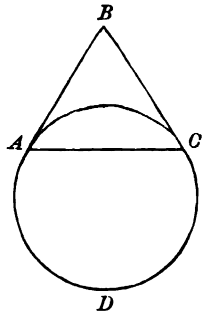


Let $B C$ be a tangent and $A B$ a chord of the circle $D A B$, intersecting at the point B; the angle $A B C$ is measured by half the arc $A B$.

At B draw the diameter $B D$. The right angle (27) $D B C$ is measured by half the arc $D A B$ (17); and the angle $D B A$ by half the arc $D A$ (23); therefore the angle $D B C — D B A$, or $A B C$, is measured by half the arc $D A B — D A$, or half the arc $A B$.

THEOREM IX.

30. *Two tangents drawn to a circumference from the same point are equal.*

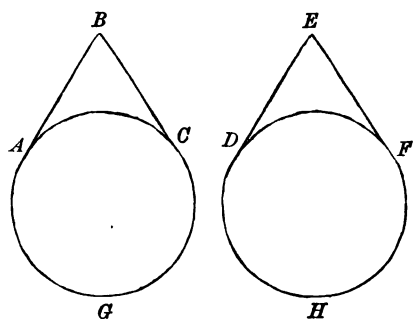


Let BA and BC be tangents from the point B to the circumference ACD; $BA = BC$. Draw the chord AC. As the angles at A and C are each measured by half the same arc (29), they are equal; hence $BA = BC$ (I. 85).

31. *Definition.* Two tangents from the same point to a circumference are called *a pair of tangents;* as BA, BC.

THEOREM X.

32. *In the same, or equal circles, pairs of tangents subtended by equal arcs are equal, and include equal angles.*



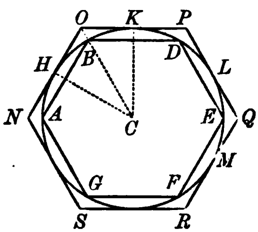
Let BA, BC, and ED, EF, be two pairs of tangents subtended by the equal arcs AC, DF, of the equal circles ACG, DFH; then $AB = BC = DE = EF$.

For, if the arc AC is placed on its equal DF, with the point A on D, the point C will fall on F, otherwise there would be points in the one or the other arc unequally distant from the centre; and the tangent AB must fall on DE, otherwise there would be at the point

D two tangents to the same circumference, that is, two perpendiculars from the same point D to the radius at that point which is impossible (I. 39). In like manner $C B$ must fall on $F E$; therefore the point B will be at E, and $A B = D E$ and $B C = E F$. But (30) $A B = B C$ and $D E = E F$; hence $A B = B C = D E = E F$. Also, as $A B$, $B C$, fall on $D E$, $E F$, the angle $B = E$.

THEOREM XI.

33. *A regular polygon inscribed in a circle being given, a similar polygon can be circumscribed about the circle.*



Let $A B D E F G$ be a regular polygon inscribed in the circle $A D F$, whose centre is C; a similar polygon can be circumscribed about $A D F$.

Bisect the arcs $A B$, $B D$, $D E$, &c., at the points H, K, L, &c., and through H, K, L, &c., draw tangents $N O$, $O P$, $P Q$, &c.; these tangents intersecting will form a circumscribed polygon $N O P Q R S$, similar to $A B D E F G$.

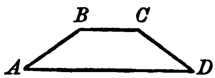
For, as the equal arcs $A B$, $B D$, $D E$, $E F$, &c., are bisected in H, K, L, M, &c., the arcs $H K$, $K L$, $L M$, &c., are equal; therefore (32) the tangent $H O = O K = K P = P L$, &c.; hence $N O = O P = P Q$, &c., and the angle $N = O = P$, &c., and the polygon $N O P Q R S$ is regular (II. 80), and as it has the same number of sides, it is similar to $A B D E F G$ (II. 81).

34. *Cor.* 1. Join $C B$. $C B$ produced will pass through O. Draw $C H$ and $C K$. As the arc $H B = B K$, $B C$ bisects the angle $H C K$; and as $H O = O K$, O must be in the line $C B$ (I. 104).

35. *Cor.* 2. If the chords $A\,H$, $H\,B$, &c., are drawn, a regular inscribed polygon will be formed of double the number of sides of $A\,B\,D\,E\,F\,G$. And if tangents are drawn through A, B, D, &c., intersecting the tangents $S\,N$, $N\,O$, $O\,P$, &c., a regular circumscribed polygon will be formed of double the number of sides of $N\,O\,P\,Q\,R\,S$.

36. *Cor.* 3. The sides of the inscribed and circumscribed polygons $A\,B\,D\,E\,F\,G$ and $N\,O\,P\,Q\,R\,S$ are parallel each to each.

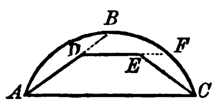
37. *Definition.* A broken line, as $A\,B\,C\,D$, is called *convex* when none of its parts produced cuts the space enclosed by the broken line and the straight line joining its extremities.



THEOREM XII.

38. *An arc of a circle is greater than any convex broken line having the same extremities which it envelops, and less than any line which envelops it and has the same extremities.*

1st. Let $A\,B\,C$ be an arc enveloping the convex broken line $A\,D\,E\,C$, both terminating at A and C; then



$$A\,B\,C > A\,D\,E\,C$$

Produce $A\,D$ and $D\,E$ till they meet the arc in B and F; then (I. 29)

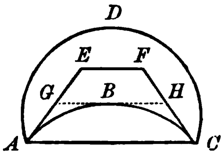
$$\text{Arc } A\,B > \text{chord } A\,B$$

$$D\,B + B\,F > D\,F$$

$$E\,F + F\,C > E\,C$$

If we add these inequalities, cancelling common parts from both members, we have

$$\text{Arc } A\,B\,C > A\,D\,E\,C$$



2d. Let $A\,B\,C$ be an arc enveloped by any lines $A\,D\,C$, $A\,E\,F\,C$, &c., all terminating at A and C; then the arc $A\,B\,C$ is the least, or shortest, of these lines.

Of the lines $A\,B\,C$, $A\,D\,C$, $A\,E\,F\,C$, &c., unless two are equal, one must be the least, or shortest line. Now, $A\,E\,F\,C$ is not the least; for drawing a tangent $G\,H$ to the arc $A\,B\,C$, as $G\,H$ is less than $G\,E\,F\,H$, $A\,G\,H\,C$ must be less than $A\,E\,F\,C$; therefore $A\,E\,F\,C$ cannot be the shortest of the enveloping lines. In like manner it can be shown that the arc $A\,D\,C$, or any other enveloping line, is not the least, or shortest line. Therefore the arc $A\,B\,C$ is the least.

39. *Corollary.* The circumference of a circle is greater than the perimeter of any polygon inscribed in it, and less than the perimeter of any polygon circumscribed about it.

DEFINITIONS.

40. A **constant quantity,** or a **constant,** is a quantity whose value does not change.

41. A **variable quantity,** or a **variable,** is a quantity whose value changes.

42. When a variable quantity changes in such a manner as constantly to approach the value of some fixed quantity, so that the difference between the variable and the constant may become less than any assignable quantity, the constant is called the *limit* of the variable.

Thus the magnitude of an angle of a regular polygon varies as the number of sides increases, and may, as the number of sides increases, approach indefinitely near to two right angles. The magnitude of the angle and the number of sides are *variables;* the two right angles is the *constant,* to which as a limit the magnitude of the angle is approaching.

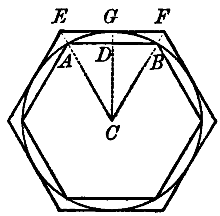
THEOREM XIII.

43. *If the number of sides of an inscribed and of a similar circumscribed regular polygon be indefinitely increased,*

1st. *The limit of the apothem of the inscribed polygon is the radius of the circle.*

2d. *The limit of the perimeters of both polygons is the circumference of the circle.*

3d. *The limit of the areas of the polygons is the area of the circle.*



1st. Let $C D$ be the apothem and $A B$ a side of a regular polygon inscribed in the circle whose radius is $C A$.

By (I. 29) $A C < C D + A D$

Hence $A C - C D < A D$

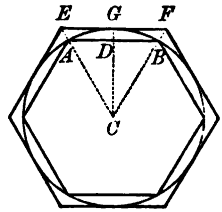
If the number of sides of the regular inscribed polygon is indefinitely increased, the length of each side will be indefinitely decreased; that is, $A B$, and much more $\frac{1}{2} A B$, or $A D$, and still more $A C - C D$, which is less than $A D$, may be made as small as we please, less than any assignable quantity; that is, the radius $C A$ is the limit of the apothem $C D$.

2d. Let $A B$ be the side of the inscribed regular polygon whose apothem is $C D$; then the tangent $E F$, parallel to $A B$ and terminating in $C A$ and $C B$ produced, will be the side of a similar circumscribed polygon (34) whose apothem will be $C G$.

Let P, p, represent respectively the perimeters of the circumscribed and inscribed polygons, and A, a, their apothems. Then by (II. 84) we have

$$P : p = A : a$$

or (II. 18) $$P - p : p = A - a : a$$



hence $$P - p = \frac{p}{a}(A - a)$$

By the first part of this proposition, by increasing the number of sides, $A - a$ may be made less than any assignable quantity; therefore $\frac{p}{a}(A - a)$, or $P - p$, will become less than any assignable quantity. But as P is always greater, and p always less (39) than the circumference of the circle, both P and p differ from the circumference less than from each other; therefore this difference may be made less than any assignable quantity. Hence, the circumference of the circle is the limit of the perimeters of both polygons.

3d. Let M, m, represent respectively the areas of the circumscribed and inscribed polygons, and P, p, their perimeters.

The triangle $E\,F\,C - A\,B\,C =$ the trapezoid $A\,E\,F\,B$
$$= \tfrac{1}{2}(EF + A\,B) \times G\,D = \tfrac{1}{2}(EF + A\,B)\,(A - a)$$
Hence the difference between the areas of the polygons,

or $$M - m = \tfrac{1}{2}(P + p)\,(A - a)$$

But by the first part of this proposition, by increasing the number of sides, $(A - a)$ may be made less than any assignable quantity; then $\frac{1}{2}(P + p)\,(A - a)$, or $M - m$, will become less than any assignable quantity. But M being always greater and m always less than the circle (39), both M and m differ less from the circle than from each other; therefore this difference may be made less than any assignable quantity. Hence the area of the circle is the limit of the areas of both polygons.

THEOREM XIV.

44. *A circle is a regular polygon of an infinite number of sides.*

By continually increasing the number of sides of a circumscribed regular polygon and a similar inscribed polygon, these may be made to differ from the circle by a quantity less than

any assignable quantity. If then we suppose that the number of sides becomes greater than any finite quantity, or infinite, then the polygons must differ from the circle by a quantity less than any finite quantity, or infinitesimal, that is, zero; therefore the circle is identical with either polygon, that is, *a circle is a regular polygon of an infinite number of sides.*

THEOREM XV.

45. *Circumferences of circles are to each other as their radii, or as their diameters; and the circles as the squares of their radii, or as the squares of their diameters.*

For circles are regular polygons of an infinite number of sides (44); and if the circumferences of circles are divided into the same infinite number of arcs, the polygons formed by their chords, that is, the circles themselves, are regular polygons of the same number of sides and are therefore similar (II. 81); and the apothems of the polygons are the radii of the circles; therefore the circumferences of the circles are as their radii (II. 84), or as twice their radii, that is, as their diameters; and the circles as the squares of their radii (II. 84) or as the squares of twice their radii, that is, as the squares of their diameters.

46. *Cor.* 1. If C and c denote the circumferences, R and r the corresponding radii, and D and d the corresponding diameters, we have

$$C : c = R : r = D : d$$

or

$$C : R = c : r$$

and

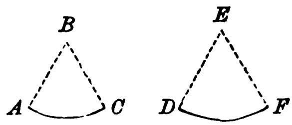
$$C : D = c : d$$

That is, the ratio of the circumference of every circle to its radius or to its diameter is the same, that is, is constant. The constant ratio of the circumference to its diameter is denoted by π (the Greek letter p).

47. *Cor.* 2.

$$\frac{C}{D} = \pi$$

$$C = \pi D = 2 \pi R$$



48. *Cor.* 3. Similar arcs, as $A C$, $D F$, are those that subtend equal angles at the centres of their respective circles; therefore similar arcs are like parts of their respective circumferences and have the same ratio as their circumferences. Similar sectors, as $A B C$, $D E F$, are also like parts of their respective circles. Therefore *similar arcs are to each other as their radii, and similar sectors as the squares of their radii.*

THEOREM XVI.

49. *The area of a circle is equal to half the product of its circumference and its radius.*

The area of a regular polygon is half the product of its perimeter and its apothem (II. 85); a circle is a regular polygon of an infinite number of sides (44); the circumference of the circle is the perimeter of the polygon, and its radius is the apothem; therefore the area of a circle is equal to half the product of its circumference and its radius.

50. *Corollary.* If $C =$ the circumference, $D =$ the diameter, $R =$ the radius, and $A =$ the area of a circle, we have

$$A = \tfrac{1}{2} C \times R$$

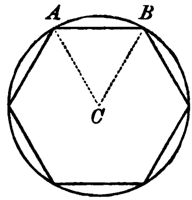
But (47) $$C = 2 \pi R = \pi D$$

Therefore $$A = \tfrac{1}{2} \times 2 \pi R \times R = \pi R^2$$

or $$A = \tfrac{1}{2} \pi D \times \frac{D}{2} = \tfrac{1}{4} \pi D^2$$

THEOREM XVII.

51. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*



In the circle whose centre is C draw the chord $A B$ equal to the radius; $A B$ is the side of a regular hexagon inscribed in the circle.

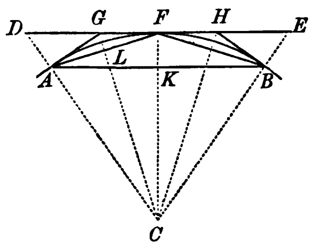
Draw the radii $C A$ and $C B$; $C A B$ is an equilateral, and therefore an equiangular triangle; hence the angle C is equal to one third of two right angles, or one sixth of four right angles; that is, the arc $A B$ is one sixth of the whole circumference, or the chord $A B$ the side of a regular hexagon inscribed in the circle (12; 26).

52. *Corollary.* The chord of half the arc $A B$ would be the side of a regular dodecagon inscribed in the circle, and the chord of one quarter of the arc $A B$, the side of a regular polygon of twenty-four sides; and so on.

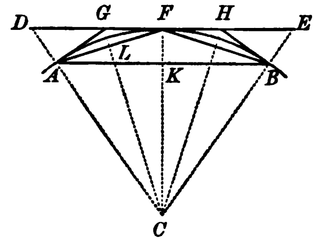
PROPOSITION XVIII.

PROBLEM.

53. *The perimeters of a regular inscribed and similar circumscribed polygon being given, to compute the perimeters of regular inscribed and circumscribed polygons of double the number of sides.*



Let $A B$ be a side of the given regular polygon inscribed in the circle whose centre is C, and $D E$ tangent at F, the middle point of the arc $A F B$, be the side of the similar circumscribed polygon. Join $A F$, and at A and B draw the tangents $A G$



and BH; then AF is a side of the regular inscribed polygon of double the number of sides, and GH a side of the similar circumscribed polygon of double the number of sides (35).

Draw CD, CG, CF, CE. CD passes through the point A, and CE through B (34); AB is parallel to DE (36).

Let P and p represent the perimeters of the given circumscribed and inscribed polygons respectively, and P' and p' respectively the perimeters of the circumscribed and inscribed polygons of double the number of sides.

From (II. 84) we have

$$P : p = CF : CK = \text{(II. 50)}\ CD : CA = CD : CF$$

And as GC bisects the angle DCF (34), by (II. 62) we have

$$CD : CF = DG : GF$$

Hence (II. 11) $$P : p = DG : GF$$

or by (II. 17, 21) $$P + p : 2p = DG + GF : 2GF = DF : GH$$

Now GH is a side of the polygon whose perimeter is P', and is contained as many times in P' as DF is in P; hence (II. 20)

$$DF : GH = P : P'$$

Therefore (II. 11) $$P + p : 2p = P : P'$$

or $$P' = \frac{2p \times P}{P + p} \qquad \{1\}$$

Again, as DE is parallel to AB, the angle $GFA = FAK$; and hence the right triangles AFK, GFL, being mutually equiangular (I. 75), are similar (II. 55), and we have

$$AK : AF = LF : GF$$

But as $A K$ is contained the same number of times in p as $A F$ is in p', we have

$$A K : A F = p : p'$$

Also as $L F$ is contained the same number of times in p' as $G F$ in P', we have

$$L F : G F = p' : P'$$

Therefore $$p : p' = p' : P'$$

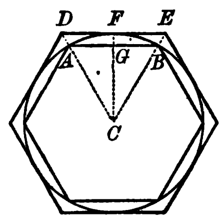
or $$p' = \sqrt{p \times P'} \qquad \{2\}$$

Therefore when p and P are given, from $\{1\}$ we can find the value of P', and then from $\{2\}$ the value of p'.

PROPOSITION XIX.

PROBLEM.

54. *The side of a regular inscribed polygon being given to compute the side of a similar circumscribed polygon.*



Let $A B$ be the given side of a regular polygon inscribed in the circle whose centre is C and $D E$ tangent at F, the middle point of the arc $A F B$, the side of a similar circumscribed polygon.

Join $C F$ and $C A$. From (II. 84) we have

$$D E : A B = F C : G C$$

Hence $$D E = \frac{A B \times F C}{G C} = \frac{A B \times R}{G C}$$

Now $G C = \sqrt{A C^2 - A G^2} = \sqrt{R^2 - \frac{1}{4} A B^2} = \frac{1}{2} \sqrt{4 R^2 - A B^2}$

Therefore $$D E = \frac{2 A B \times R}{\sqrt{4 R^2 - A B^2}}$$

55. *Corollary.* If $A\,B$ is the side of a regular inscribed hexagon and the diameter is unity, then (51) $A\,B = R = \frac{1}{2}$, and

$$D\,E = \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}$$

and the perimeter of the regular circumscribed hexagon $= 2\sqrt{3} = 3.4641016$.

PROPOSITION XX.

PROBLEM.

56. *To find the arithmetical value of the constant* π.

From (47) $C = \pi\,D$; if $D = 1$, this equation becomes $C = \pi$. If then we can compute the circumference of a circle whose diameter is unity, we shall have the value of π.

If the diameter is unity, radius is one half, and the side of a regular hexagon inscribed in the circle is one half (51), and the perimeter of the hexagon is $6 \times \frac{1}{2} = 3$, and the perimeter of the regular circumscribed hexagon (55) is 3.4641016.

Now by (53) we can compute the perimeters of the regular circumscribed and inscribed dodecagons; for taking

$$P = 3.4641016$$

and

$$p = 3.0000000$$

$$P' = \frac{2\,p \times P}{P + p} = 3.2153904$$

and

$$p' = \sqrt{p \times P'} = 3.1058285$$

Then, taking the dodecagons as the given polygons, we can compute the perimeters of the regular circumscribed and inscribed polygons of 24 sides. Thus, taking

$$P = 3.2153904 \qquad \text{and } p = 3.1058285$$

we find by the same formulas that

$$P' = 3.1596602 \qquad \text{and } p' = 3.1326286$$

Continuing this process we obtain the results given in the following table: —

No. of sides.	Perimeter of Circumscribed Polygon.	Perimeter of Inscribed Polygon.
6	3.4641016	3.0000000
12	3.2153904	3.1058285
24	3.1596602	3.1326286
48	3.1460863	3.1393502
96	3.1427106	3.1410319
192	3.1418712	3.1414525
384	3.1416616	3.1415576
768	3.1416092	3.1415839
1536	3.1415963	3.1415904
3072	3.1415929	3.1415920

Now the circumference of the circle is less than the perimeter of the circumscribed polygon and greater than the perimeter of the inscribed polygon (39); that is, less than 3.1415929, but greater than 3.1415920. It follows therefore that as far as the sixth decimal place

$$\pi = 3.141592$$

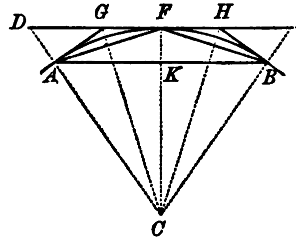
By more expeditious methods the value of π has been calculated to over 200 places of decimals. The value to 208 decimal places is

3.14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679
82148086513282306647093844609550582231725359408128
48473781392038633830215747399600825931259129401839
80651744

PROPOSITION XXI.

PROBLEM.

57. *The areas of a regular inscribed and similar circumscribed polygon being given, to compute the areas of the regular inscribed and circumscribed polygons of double the number of sides.*



Let $A\,B$ be a side of the given regular polygon inscribed in a circle whose centre is C, and $D\,E$ tangent at F, the middle point of the arc $A\,F\,B$, a side of the similar circumscribed polygon.

Join $A\,F$, and at A and B draw the tangents $A\,G$ and $B\,H$; then $A\,F$ is a side of the regular inscribed polygon of double the number of sides, and $G\,H$ a side of the similar circumscribed polygon of double the number of sides (35).

Let P and p represent respectively the areas of the given circumscribed and inscribed polygons, and P' and p' respectively the areas of the circumscribed and inscribed polygons of double the number of sides. As the triangles $C\,A\,K$, $C\,A\,F$, have the same vertex, A, and their bases in the same straight line, we have (II. 47)

$$C\,A\,K : C\,A\,F = C\,K : C\,F$$

But $C\,A\,K$ is contained the same number of times in p as $C\,A\,F$ is in p'; that is

$$C\,A\,K : C\,A\,F = p : p'$$

Hence $$C\,K : C\,F = p : p'$$

And as the triangles $C\,A\,F$, $C\,D\,F$, have the same vertex, F, and their bases in the same straight line, we have (II. 47)

$$C\,A\,F : C\,D\,F = C\,A : C\,D$$

But $C\,A\,F$ is contained the same number of times in p' as $C\,D\,F$ is in P; that is

$$C\,A\,F : C\,D\,F = p' : P$$

Hence $$C\,A : C\,D = p' : P$$

But as AB is parallel to DE, we have (II. 50)

$$CK : CF = CA : CD$$

Therefore $$p : p' = p' : P$$

or $$p' = \sqrt{p \times P} \qquad \{1\}$$

Again, as the triangles CGF, CDG, have the same vertex, C, and their bases in the same straight line, we have (II. 47)

$$CGF : CDG = GF : DG = \text{(II. 62) } CF : CD$$

But (II. 50) $CF : CD = CK : CA = CK : CF = p : p'$

Hence $$CGF : CDG = p : p'$$

and (II. 21, 17) $2\,CGF : CGF + CDG = 2p : p + p'$

or $$CAGF : CDF = 2p : p + p'$$

But $CAGF$ is contained the same number of times in P' as CDF is in P; that is

$$CAGF : CDF = P' : P$$

Therefore $$2p : p + p' = P' : P$$

or $$P' = \frac{2p \times P}{p + p'} \qquad \{2\}$$

Therefore when p and P are given, from $\{1\}$ we can find the value of p', and then from $\{2\}$ the value of P'.

58. *Scholium.* This furnishes another method of finding the value of π.

PROPOSITION XXII.

PROBLEM.

59. *To find the arithmetical value of the constant π.*

From (50) $A = \pi R^2$; if $R = 1$, this equation becomes $A = \pi$. If then we can compute the area of a circle whose radius is unity we shall have the value of π.

If the radius is unity, the diameter is 2, and the area of the circumscribed square is 4, and the area of the inscribed square, being half of the circumscribed square, is 2.

Now by (57) we can find the area of the regular inscribed and circumscribed octagons; for taking

$$p = 2 \qquad \text{and } P = 4$$

$$p' = \sqrt{p \times P} = \sqrt{8} = 2.8284271$$

and

$$P' = \frac{2p \times P}{p + p'} = \frac{16}{2 + \sqrt{8}} = 3.3137085$$

Then taking the octagons as the given polygons, we can find the areas of the regular inscribed and circumscribed polygons of 16 sides; for, now taking

$$p = 2.8284271 \qquad \text{and } P = 3.3137085$$

we find by the same formulas that

$$p' = 3.0614675 \qquad \text{and } P' = 3.1825979$$

Continuing this process we obtain the results given in the following table:—

No. of sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614675	3.1825979
32	3.1214452	3.1517249
64	3.1365485	3.1441184
128	3.1403312	3.1422236
256	3.1412773	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928

Now the area of the circle is greater than the area of the inscribed polygon and less than the area of the circumscribed polygon (39); that is, greater than 3.1415923 and less than 3.1415928. It follows therefore that as far as the sixth decimal place

$$\pi = 3.141592$$

60. *Scholium.* We can also begin with the regular inscribed hexagon, whose perimeter, if $R = 1$, is 6 (51), and whose apothem is $\sqrt{1^2 - (\frac{1}{2})^2} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$; therefore the area of the inscribed hexagon (II. 85) $= \frac{3}{2}\sqrt{3} = 2.5980762$.

If p represent the area of the inscribed regular hexagon, P the similar circumscribed hexagon, a and A their apothems respectively, by (II. 84) we have

$$a^2 : A^2 = p : P, \quad \text{or } (\tfrac{1}{2}\sqrt{3})^2 : 1^2 = \tfrac{3}{2}\sqrt{3} : P$$

Hence $\qquad P = 2\sqrt{3} = 3.4641016$

Applying the formulas, $p' = \sqrt{p \times P}$, and $P' = \dfrac{2p \times P}{p + p'}$, we obtain the results given in the following table: —

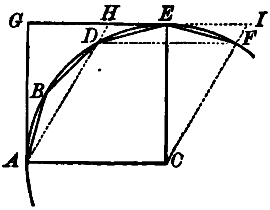
No. of sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
6	2.5980762	3.4641016
12	3.0000000	3.2153904
24	3.1058285	3.1596602
48	3.1326286	3.1460863
96	3.1393502	3.1427106
192	3.1410319	3.1418712
384	3.1414525	3.1416616
768	3.1415576	3.1416092
1536	3.1415839	3.1415963
3072	3.1415904	3.1415929

61. *Corollary.* From this last table it appears that the area of the inscribed regular dodecagon is exactly three times the square of the radius. This can also be proved by direct geometrical proof.

PROPOSITION XXIII.

THEOREM.

62. *The area of a regular dodecagon inscribed in a circle is equal to three times the square described on the radius of the circle.*



Let $A\,B\,D\,E\,F$, &c., be a regular dodecagon inscribed in the circle whose centre is C; then the dodecagon $A\,B\,D\,E\,F$, &c., is equal to three times the square described on the radius $A\,C$.

Draw $A\,D$, $D\,F$, $C\,E$, $C\,F$; through E draw the tangent $G\,I$ meeting $A\,D$ and $C\,F$ produced in H and I; and through A draw the tangent $A\,G$ completing the figure $A\,G\,E\,C$.

As the arc $A\,B = B\,D = D\,E = E\,F = \frac{1}{12}$ of the whole circumference of the circle, the arc $A\,F$ is a third, $A\,E$ a quarter, and $A\,D$ a sixth, of the whole circumference; therefore the figure $A\,B\,D\,E\,F\,C$ is a third of the inscribed dodecagon, the angle $E\,C\,A$ a right angle, and $A\,D$ and $D\,F$ each equal to $A\,C$ or $C\,F$ (51). Therefore $A\,G\,E\,C$ is the square described on the radius, and $A\,D\,F\,C$ a parallelogram (I. 121); and as $H\,I$ is parallel to $A\,C$, $A\,H\,I\,C$ and $D\,H\,I\,F$ are also parallelograms.

Now the triangle $A\,B\,D = D\,E\,F$ (I. 88), and $D\,H\,I\,F = 2\,D\,E\,F$ (II. 43, 45); therefore $D\,H\,I\,F = A\,B\,D + D\,E\,F$; adding to each member of this equation $A\,D\,F\,C$ we have

$$A\,H\,I\,C = A\,B\,D\,E\,F\,C$$

But (II. 44) $\qquad A\,H\,I\,C = A\,G\,E\,C$

therefore $\qquad A\,B\,D\,E\,F\,C = A\,G\,E\,C$

that is, the area of one third of the inscribed dodecagon is equal to the square described on the radius; therefore the area of the dodecagon itself is equal to three times the square described on the radius.

PRACTICAL QUESTIONS.

1. What is the circumference of a circle whose radius is 10 feet?

2. What is the diameter of a circle whose circumference is 57 rods?

3. What is the area of a circle whose radius is 40 feet?

4. What is the area of a circle whose circumference is 18 inches?

5. What is the circumference of a circle whose area is 116 square feet?

6. The radii of two concentric circles are 40 and 54 feet; what is the area of the space bounded by their circumferences?

7. A has a circular lot of land whose diameter is 95 rods, and B a similar lot whose area is 750 square rods; compare these lots.

8. What is the difference between the perimeters of two lots of land each containing an acre, if one is a square and the other a circle?

9. What is the area of a square inscribed in a circle whose area is a square metre?

10. What is the area of a regular hexagon inscribed in a circle whose area is 567 square feet.

11. If a rope an inch in diameter will support 1,000 pounds, what must be the diameter of a rope of like material to support 4,000 pounds?

12. If a pipe an inch in diameter will fill a cistern in 25 minutes, how long will it take a pipe 5 inches in diameter?

13. If a pipe an inch in diameter will empty a cistern in an hour, how long will it take this pipe to empty the cistern if there is another pipe one third of an inch in diameter through which the fluid runs in?

Ans. $67\frac{1}{2}$ minutes.

14. If a pipe 3 inches in diameter will empty a cistern in 3 hours, how long will it take the pipe to empty the cistern if there are 3 other pipes each an inch in diameter through which the fluid runs in.

Ans. $4\frac{1}{2}$ hours.

EXERCISES.

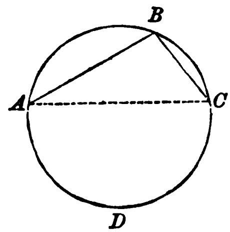
The following Theorems, depending for their demonstration upon those already demonstrated, are introduced as exercises for the pupil. In some of them references are made to the propositions upon which the demonstration depends. They are not connected with the propositions in the following books, and can be omitted if thought best.

63. Every diameter bisects the circle and the circumference.

64. A straight line can meet the circumference of a circle in only two points. (6.) (I. 90.)

65. The diameter is greater than any other chord of the circle.

66. In the same or equal circles, when the sum of the arcs is less than a circumference, the greater arc is subtended by the greater chord; and, conversely, the greater chord is subtended by the greater arc.



Draw $A\,C$. (23.) (I. 87.)

What is the case when the sum of the arcs is greater than a circumference?

67. Equal chords are equally distant from the centre; and of two unequal chords the greater is nearer the centre.

68. The shortest and the longest line that can be drawn from any point to a given circumference lies on the line that passes from the point to the centre of the circle.

69. Two parallels cutting the circumference of a circle intercept equal arcs.

70. (Converse of 69.) Two lines intercepting equal arcs of a circumference, if they do not intersect each other within the circle, are parallel.

71. The lines joining the extremities of two diameters are parallel.

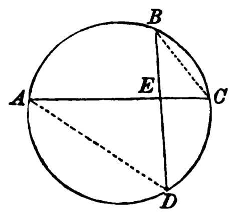
72. If the extremities of two intersecting chords are joined, the opposite, or vertical, triangles thus formed are similar.

73. If two circumferences cut each other, the chord which joins their points of intersection is bisected at right angles by the line joining their centres. (14.)

74. If two circumferences touch each other, their centres and point of contact are in the same straight line, perpendicular to the tangent at the point of contact. (27.)

75. The distance between the centres of two circles whose circumferences cut one another is less than the sum, but greater than the difference, of their radii.

76. Every angle inscribed in a segment greater than a semicircle is acute; and every angle inscribed in a segment less than a semicircle is obtuse. (23.)



77. The angle formed by two chords cutting each other within the circle is measured by half the sum of the arcs intercepted by its sides and by the sides of its vertical angle.

Join $B\,C$. (23.)

78. By moving the point of intersection of the two chords, show that (16) and (23) can be deduced from (77).

79. The segments of two chords cutting each other within a circle are reciprocally proportional; that is, $A\,E : B\,E = E\,D : E\,C$.

Join $A\,D$, $B\,C$. (72.) (II. 55.)

80. The opposite angles of a quadrilateral inscribed in a circle are supplementary. (23.)

81. A quadrilateral whose opposite angles are supplementary, and no other, can have a circle circumscribed about it.

82. The sum of the opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides. (30.)

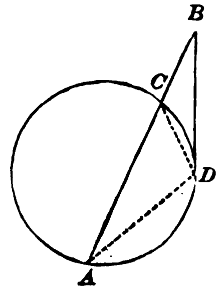
83. The area of a sector is equal to half the product of its arc by the radius of the circle. (49.)

84. Show how to find the area of a segment of a circle.

85. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the circle.

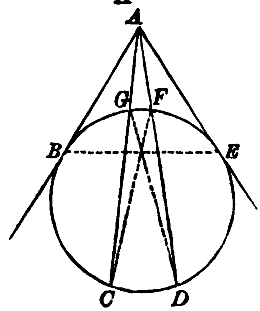
86. A tangent is a mean proportional between a secant drawn from the same point and the part of the secant without the circle.

Join $A\,D$, $D\,C$. (29, 23.) (II. 112.)



87. The angle formed by two secants, two tangents, or a secant and a tangent cutting each other without the circle, is measured by half the difference of the intercepted arcs.

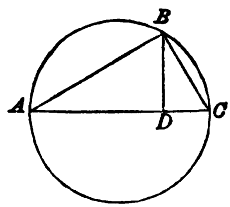
Join $C\,F$. (I. 79.) (23.)



88. By moving the point of intersection, show that (23) can be deduced from (87). Show also that (69) can be deduced from (87).

89. Two secants drawn from the same point are to each other inversely as the parts of the secants without the circle.

Join $C\,F$, $D\,G$. (23.) (II. 112.)



90. A perpendicular from a circumference to the diameter is a mean proportional between the segments of the diameter.

Join $A\,B$, $B\,C$. (25.) (II. 65.)

91. If from one end of a chord a diameter is drawn, and from the other end a perpendicular to this diameter, the chord is a mean proportional between the diameter and the adjacent segment of the diameter.

Join $A\,B$. (25.) (II. 64.)

92. From the tables in (56) and (60) it will be seen that the perimeters of the regular circumscribed polygons, when the diameter is unity, is numerically the same as their areas when the radius is unity; but the perimeters of the inscribed regular polygons when the diameter is unity is numerically the same as the areas of the regular inscribed polygons of double the number of sides when the radius is unity. Discuss the subject.

BOOK IV.

MAXIMA AND MINIMA OF PLANE FIGURES.

ISOPERIMETRICAL FIGURES.

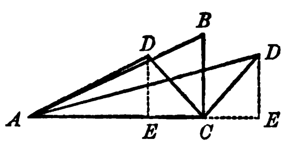
DEFINITIONS.

1. Of quantities of the same kind that which is greatest is called a *maximum;* that which is least a *minimum.*

2. **Isoperimetrical Figures** are those whose perimeters are equal.

THEOREM I.

3. *Of all triangles formed with two given sides that formed with these two sides at right angles to each other is the maximum.*

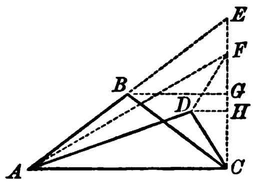


Let the triangle $A\,B\,C$ have the angle at C a right angle, and the two sides $A\,C$, $C\,B$ equal respectively to $A\,C$, $C\,D$ of the triangle $A\,D\,C$; the triangle $A\,B\,C > A\,D\,C$.

Draw $D\,E$ perpendicular to $A\,C$; $D\,E$ is the altitude of $A\,D\,C$, and $B\,C$ of $A\,B\,C$. The triangles $A\,B\,C$, $A\,D\,C$ having the same base $A\,C$, are to each other as their altitudes (II. 47). But (I. 90) $D\,E < D\,C = B\,C$. Hence triangle $A\,B\,C > A\,D\,C$.

THEOREM II.

4. *Of isoperimetrical triangles having the same base the isosceles is the maximum.*



Of the isoperimetrical triangles $A B C$, $A D C$ having the same base $A C$, let $A B C$ be isosceles;

then $\quad A B C > A D C$

Produce $A B$ to E, making $B E = A B$; join $E C$. Since $B E = B A = B C$, a semicircumference described on $A E$ as a diameter will pass through C; hence $A C E$ is a right angle (III. 25).

Draw $D F = D C$.

$$A D + D F = (A D + D C = A B + B C = A B + B E =) A E$$

Hence $A D$, $D F$ cannot be in the same straight line, unless $A D$, $D F$ fall on $A E$ (I. 92); in this case, as $D F = D C$, D would fall on B, and $A D C$ would be isosceles, which is contrary to the hypothesis.

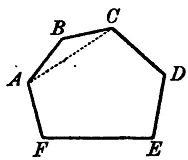
Through B and D draw $B G$, $D H$ parallel to $A C$. $B G$, $D H$ are perpendicular to $E C$ (I. 54), and as $B E C$ and $D F C$ are isosceles triangles, $G C = \frac{1}{2} E C$, and $H C = \frac{1}{2} F C$ (I. 83).

Draw $A F$. $A F < A D + D F = A E$; hence (I. 90, converse of 3d part) $F C < E C$, and $H C < G C$.

As $A B C$, $A D C$ have the same base $A C$, they are to each other as their altitudes $G C$, $H C$ (II. 47); hence as $G C > H C$, the triangle $A B C > A D C$.

THEOREM III.

5. *Of isoperimetrical polygons of the same number of sides the maximum is equilateral.*

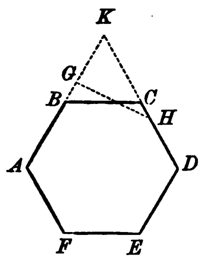


Let $A\,B\,C\,D\,E\,F$ be a polygon; to be the maximum $A\,B = B\,C = C\,D = D\,E$, &c.

Join $A\,C$. If $A\,B$ is not equal to $B\,C$, another triangle with the same perimeter and a greater area can be substituted for $A\,B\,C$ (4), and the area of the whole polygon be increased without increasing the perimeter of the polygon. Hence to be a maximum $A\,B = B\,C = C\,D = D\,E$, &c.

THEOREM IV.

6. *Of equilateral isoperimetrical polygons of the same number of sides the maximum is equiangular.*



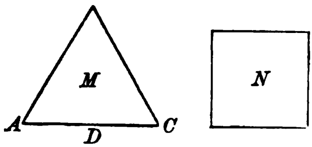
Let $A\,D$ be an equilateral polygon; to be the maximum the angle $A = B = C = D$, &c.

Produce $A\,B$, $D\,C$ till they meet in K. If the angle $A\,B\,C$ is not equal to $B\,C\,D$, one must be greater; suppose $A\,B\,C > B\,C\,D$; then the angle $K\,C\,B > K\,B\,C$, and (I. 87) $K\,B > K\,C$. Cut off $K\,G = K\,C$ and $K\,H = K\,B$, and join $G\,H$; the two triangles $B\,K\,C$, $G\,K\,H$ are equal (I. 80), and hence the polygon $A\,B\,C\,D\,E\,F$ is equivalent to $A\,G\,H\,D\,E\,F$. Hence, as $A\,B\,C\,D\,E\,F$ is a maximum polygon, $A\,G\,H\,D\,E\,F$ must also be a maximum polygon, and must therefore be equilateral (5), that is, $A\,G = G\,H = H\,D = D\,E$, &c.; but by hypothesis $A\,B\,C\,D\,E\,F$ is equilateral, that is, $A\,B = B\,C = C\,D = D\,E$, &c. Hence $A\,B = A\,G$, which is absurd. Hence the angle $A\,B\,C$ must be equal to $B\,C\,D$, and the polygon be equiangular.

7. *Corollary.* Of isoperimetrical polygons of the same number of sides the maximum polygon is equilateral (5) and equiangular (6), that is, is regular. And, *conversely*, of isoperimetrical polygons of the same number of sides the regular polygon is the maximum.

THEOREM V.

8. *Of isoperimetrical regular polygons that which has the greatest number of sides is the maximum.*

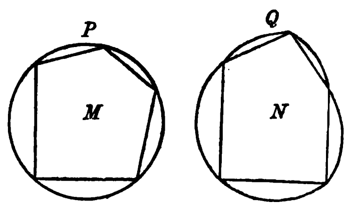


Let M be a regular polygon of three sides, and N the isoperimetrical regular polygon of four sides; then $M < N$. For, take any point D in AC, the polygon M can be considered an irregular polygon of four sides, the angle D being equal to two right angles. But the irregular polygon M of four sides is less than the isoperimetrical regular polygon N of four sides (7). In the same manner it can be proved that the regular polygon N of four sides is less than the isoperimetrical regular polygon of five sides, and so on.

9. *Corollary.* Of isoperimetrical polygons, the circle (III. 44) is the maximum.

THEOREM VI.

10. *Of polygons formed of given sides the one that can be inscribed in a circle is the maximum.*

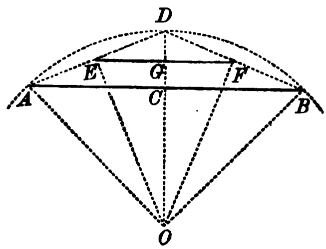


Let M be a polygon inscribed in a circle P, and N be a polygon with the same sides which cannot be inscribed in a circle; then $M > N$. On the several sides of N describe segments equal to the segments on the homologous sides of M, thus forming the irregular plane figure Q having the same perimeter as P. The circle P is greater than the isoperimetrical figure Q (9); subtracting the equal segments from each, we have the polygon $M > N$.

PROPOSITION VII.

PROBLEM.

11. *The radius and apothem of a regular polygon being given, to compute the radius and apothem of the isoperimetrical regular polygon of double the number of sides.*



Let $A\,B$ be the side, O the centre, $O\,A$ the radius (II. 83), and $O\,C$ the apothem of the given polygon.

Produce $O\,C$ to D, a point in the circumference circumscribed about the polygon. Join $A\,D$, $D\,B$; bisect $A\,D$, $D\,B$ in E, F, and draw $E\,F$, $O\,E$, and $O\,F$. As the side $E\,F$ is parallel to $A\,B$ (II. 51), the triangles $E\,D\,F$ and $A\,D\,B$ are mutually equiangular, and hence similar (II. 55); therefore $E\,F$ is equal to one half $A\,B$ (II. 52), and is the side of a regular polygon of the same perimeter and of double the number of sides as the given polygon. As the required polygon has twice as many sides as the given polygon, the angle at its centre included by two of its adjacent radii must be one half of $A\,O\,B$; but as $E\,O$ and $F\,O$ respectively bisect the angles $A\,O\,D$ and $D\,O\,B$ (I. 83), the angle $E\,O\,F$ is one half of $A\,O\,B$; therefore O is also the centre of the required polygon; and $O\,E$ is the radius, and $O\,G$ the apothem.

Put $\qquad r = O\,A,\ a = O\,C,\ r' = O\,E,\ a' = O\,G$

Now the point G is in the middle of $D\,C$ (II. 50);

hence $\qquad O\,G = \frac{1}{2}\,(O\,C + O\,D)$

or $\qquad a' = \frac{1}{2}\,(a + r) \qquad \{1\}$

Also, as OED is a right triangle, we have (II. 64)

$$OE^2 = OG \times OD$$

or

$$r' = \sqrt{a' \times r} \qquad \{2\}$$

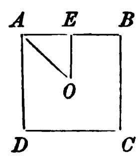
12. *Cor.* 1. $OA > OE$, and $OC < OG$; that is, of a regular polygon the radius is greater and the apothem less than the radius and apothem of the isoperimetrical polygon of double the number of sides; hence the difference between the radius and apothem diminishes as the number of sides increases. By continually doubling the number of sides this difference may be made as small as we please, or zero, when the polygon has an infinite number of sides, i. e. when it becomes a circle.

13. *Cor.* 2. This problem furnishes another method of finding the arithmetical value of the constant π.

From (III. 47) $C = 2\pi R$; if we put the circumference $C = 1$, this equation becomes

$$1 = 2\pi R, \text{ or } \pi = \frac{1}{2R}$$

If then we can find the radius of a circle whose circumference is unity, we can find the value of π.



Let $ABCD$ be a square whose perimeter $= 1$; then $AB = \frac{1}{4}$. Let r denote its radius, OA, and a its apothem, OE; then we have

$$a = \tfrac{1}{8} = 0.1250000$$

$$r = \tfrac{1}{8}\sqrt{2} = 0.1767767$$

From $\{1\}$ and $\{2\}$ we can compute a' and r' of the regular octagon of the same perimeter as the square.

$$a' = \tfrac{1}{2}(a + r) = 0.1508883$$

$$r' = \sqrt{a' \times r} = 0.1633203$$

Then taking the apothem and radius of the octagon as a and r,

or $\qquad a = 0.1508883 \qquad r = 0.1633203$

we find by the same formulas the apothem and radius of the isoperimetrical regular polygon of 16 sides,

$$a' = 0.1571048 \qquad r' = 0.1601822$$

Continuing this process we obtain the results given in the following table:

No. of sides.	Apothems.	Radii.
4	0.1250000	0.1767767
8	0.1508883	0.1633203
16	0.1571048	0.1601822
32	0.1586433	0.1594109
64	0.1590270	0.1592188
128	0.1591230	0.1591709
256	0.1591469	0.1591589
512	0.1591529	0.1591559
1024	0.1591544	0.1591552
2048	0.1591548	0.1591550
4096	0.1591549	0.1591549

Now a circumference described with the radius a is inscribed in the polygon, and a circumference described with the radius r is circumscribed about the polygon. Hence the circle isoperimetrical with the polygon has a radius always greater than a and less than r. But a and r of the polygon of 4096 sides do not differ so much as 0.0000001; therefore the radius of the circumference which is equal to the perimeter of the polygons, that is, to 1, is 0.1591549 within less than 0.0000001.

Hence $\qquad \pi = \dfrac{1}{2 \times 0.1591549} = 3.141593$

within 0.000001.

EXERCISES.

14. If from two points on the same side of a straight line straight lines are drawn to a given point of this line, the sum of these lines is a minimum when they make equal angles with the given line. (I. 48, 81, 29.)

15. Of triangles having equal angles at the vertex and equal bases, the isosceles has the maximum area (III. 24; II. 47) and the maximum perimeter. (14.)

16. Of isoperimetrical triangles having equal bases the isosceles not only has the maximum area (4), but also the maximum angle at the vertex. (15, 2d part; I. 136.)

17. Of triangles having the same base and altitude the isosceles has the minimum perimeter (14), and at the vertex the maximum angle.

18. Of triangles having the sum of two sides and their included angle equal the isosceles has the maximum area and the minimum base.

19. Of triangles having the sum of two sides and their included angle equal that in which the difference of the sides is the maximum is the minimum, and its base is the maximum. (18.)

20. Of equivalent triangles having equal angles at the vertex the isosceles has the minimum perimeter. (18.)

21. Of isoperimetrical triangles having equal angles at the vertex, the isosceles has the maximum area and the minimum base.

22. Of equivalent polygons of the same number of sides the regular polygon has the minimum perimeter. (7.)

23. Of equivalent regular polygons that which has the maximum number of sides has the minimum perimeter. (8.)

24. To cover a pavement with blocks of a given area, what must be the shape of the blocks that the extent of the lines between the blocks shall be a minimum?

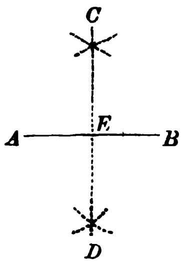
BOOK V.

PROBLEMS OF CONSTRUCTION.

IN the preceding demonstrations we have assumed that our figures were already constructed. The Problems of Construction given in this Book depend for their solution upon the principles of the preceding Books. In some of the problems the construction and demonstration are given in full; in others the construction is given and the propositions necessary to prove the construction referred to in the order in which they are to be used, and the pupil must complete the demonstration. In a few instances references are made to the Exercises appended to the previous Books. In such cases either the propositions to which reference is made can be demonstrated or the problem omitted.

PROBLEM I.

1. *To bisect a given straight line.*



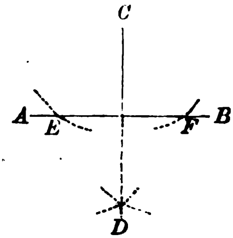
Let $A B$ be the given straight line. From A and B as centres with a radius greater than half of $A B$, describe arcs cutting one another at C and D; join C and D cutting $A B$ at E, and the line $A B$ is bisected at E. For C and D being each equally distant from A and B, the line $C D$ must be perpendicular to $A B$ at its middle point (converse of I. 94).

PROBLEM II.

2. *From a given point without a straight line to draw a perpendicular to that line.*

Let C be the point and AB the line.

From C as a centre describe an arc cutting AB in two points E and F; with E and F as centres, with a radius greater than half EF, describe arcs intersecting at D. Draw CD, and it is the perpendicular required (converse of I. 94).

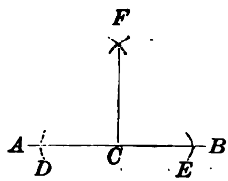


PROBLEM III.

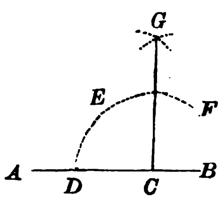
3. *From a given point in a straight line to erect a perpendicular to that line.*

Let C be the given point and AB the given line.

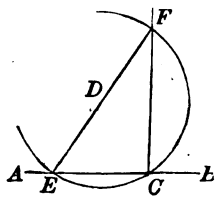
With C as a centre describe an arc cutting AB in D and E; with D and E as centres, with a radius greater than DC, describe arcs intersecting at F. Draw CF, and it is the perpendicular required (converse of I. 94).



Second Method. With C as a centre describe an arc DEF; take the distances DE and EF equal to CD, and from E and F as centres, with a radius greater than half the distance from E to F, describe arcs intersecting at G. Draw CG, and it is the perpendicular required (III. 51; III. 13; III. 17).



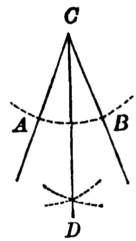
Third Method. With any point, D, without the line $A\,B$, with a radius equal to the distance from D to C, describe an arc cutting $A\,B$ at E; draw the diameter $E\,D\,F$. Draw $C\,F$, and it is the perpendicular required (III. 25).



PROBLEM IV.

4. *To bisect a given arc, or angle.*

1st. Let $A\,B$ be the given arc. Draw the chord $A\,B$ and bisect it with a perpendicular (1; III. 13).

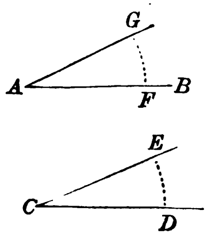


2d. Let C be the given angle.

With C as a centre describe an arc cutting the sides of the angle in A and B; bisect the arc $A\,B$ with the line $C\,D$, and it will also bisect the angle C (III. 11).

PROBLEM V.

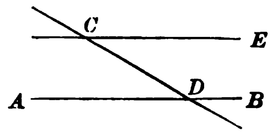
5. *At a given point in a straight line to make an angle equal to a given angle.*



Let A be the given point in the line $A\,B$, and C the given angle. With C as a centre describe an arc $D\,E$ cutting the sides of the angle C; with A as a centre, with the same radius, describe an arc; with F as a centre, with a radius equal to the distance from D to E, describe an arc cutting the arc $F\,G$. Draw $A\,G$. The angle $A = C$ (III. 12; III. 11).

PROBLEM VI.

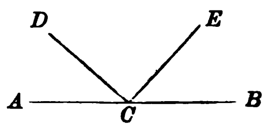
6. *Through a given point to draw a line parallel to a given straight line.*



Let C be the given point, and $A\,B$ the given line. From C draw a line $C\,D$ to $A\,B$; at C in the line $D\,C$ make an angle $D\,C\,E$ equal to $C\,D\,A$ (5); $C\,E$ is parallel to $A\,B$ (I. 56).

PROBLEM VII.

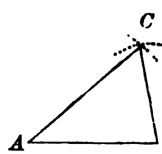
7. *Two angles of a triangle given, to find the third.*



Draw an indefinite line $A B$; at any point C make an angle $A\,C\,D$ equal to one of the given angles, and $D\,C\,E$ equal to the other (5). Then $E\,C\,B$ is the third angle (I. 44; I. 73).

PROBLEM VIII.

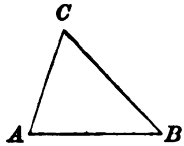
8. *The three sides of a triangle given, to construct the triangle.*



Take $A\,B$ equal to one of the given sides; with A as a centre, with a radius equal to another of the given sides, describe an arc, and with B as a centre, with a radius equal to the remaining side, describe an arc intersecting the first arc at C. Draw $A\,C$ and $C\,B$, and $A\,C\,B$ is evidently the triangle required.

PROBLEM IX.

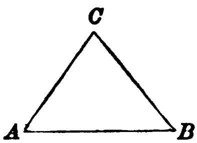
9. *Two sides and the included angle of a triangle given, to construct the triangle.*



Draw $A\,B$ equal to one of the given sides; at B make the angle $A\,B\,C$ equal to the given angle (5), and take $B\,C$ equal to the other given side; join A and C, and $A\,B\,C$ is evidently the triangle required.

PROBLEM X.

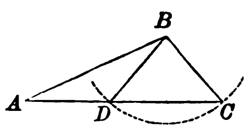
10. *Two angles and a side of a triangle given, to construct the triangle.*



If the angles given are not both adjacent to the given side, find the third angle by (7). Then draw $A\,B$ equal to the given side, and at B make an angle $A\,B\,C$ equal to one of the angles adjacent to $A\,B$, and at A make an angle $B\,A\,C$ equal to the other angle adjacent to $A\,B$, and $A\,B\,C$ is evidently the triangle required.

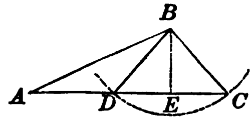
PROBLEM XI.

11. *Two sides of a triangle and the angle opposite one of them given, to construct the triangle.*



Draw an indefinite line $A\,C$; at A make the angle $C\,A\,B$ equal to the given angle, and take $A\,B$ equal to the side adjacent to the given angle; with B as a centre, with a radius equal to the other given side, describe an arc cutting $A\,C$. If the given angle A is acute,

1st. The given side BC, opposite the given angle, may be less than the other given side; then the arc described from B as a centre will cut AC in two points, C and D, on the same side of A, and, drawing



BC and BD, the triangles ABC and ABD (whose angle BDA is the supplement of the angle BCA), both satisfy the given conditions.

2d. The given side opposite the given angle may be equal to the perpendicular BE; then the arc described from B as a centre will touch AC, and the right triangle ABE is the only one that can satisfy the given conditions.

3d. The side opposite the given angle may be greater than the other given side; then the arc described from B as a centre will cut AC in C, and in another point on the other side of A. In this case there can be but one triangle ABC satisfying the given conditions, the triangle formed on the opposite side of AB containing not the given angle but its supplement.

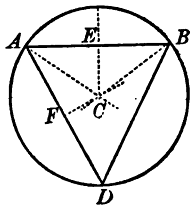
4th. If the given angle is not acute, the given side opposite the given angle must be greater than the other given side, and, as in the last case above, there can be but one solution.

12. *Scholium* 1. If the side opposite the given angle A is less than the perpendicular, or if the given angle is not acute, and at the same time the side opposite the given angle is less than the other given side, the solution is impossible.

13. *Scholium* 2. Compare with this (I. 96 – 100).

PROBLEM XII.

14. *To describe a circle about a given triangle.*



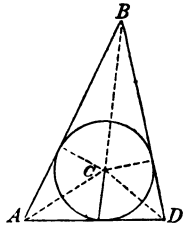
Let $A\,B\,D$ be the given triangle. Bisect the two sides $A\,B$ and $A\,D$ at E and F; from the points E and F draw $E\,C$ and $F\,C$ perpendicular respectively to $A\,B$ and $A\,D$. With the point C, where $E\,C$ and $F\,C$ intersect, as a centre, with a radius equal to the distance of C from any of the vertices, describe a circle, and it will be the circle required (I. 108).

15. *Scholium* 1. A circumference may be made to pass through any three given points, as A, B, D, by drawing $A\,B$, $A\,D$, and proceeding as in (14). If the three points, A, B, D, are in the same straight line, the perpendiculars bisecting $A\,B$, $A\,D$, will be parallel (I. 56), and will never meet; that is, a straight line may be considered the circumference of a circle whose radius is infinity.

16. *Scholium* 2. The centre of a given circumference, or of a given arc, can be found by taking any three points in the circumference, as A, B, D, and proceeding as in (15). Or, any two chords, not parallel, can be drawn, and bisected by perpendiculars; where these perpendiculars intersect will be the centre of the given circumference or arc (III. 14, 93).

PROBLEM XIII.

17. *To inscribe a circle in a given triangle.*



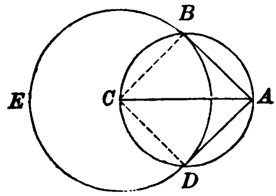
Let $A\,B\,D$ be the given triangle. Bisect the angles A and B with the lines $A\,C$ and $B\,C$. With the point C, where $A\,C$ and $B\,C$ intersect, as a centre, with a radius equal to the distance from C to any of the sides, describe a circle, and it will be the circle required (I. 106).

PROBLEM XIV.

18. *Through a given point to draw a tangent to a given circumference.*

1st. If the given point is in the circumference.

Erect a perpendicular (3) to the radius at the given point (III. 28).

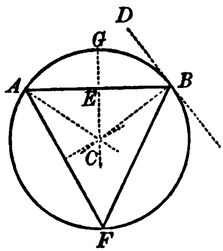


2d. If the given point is without the circumference.

Join the given point A with the centre C of the given circle $B\,D\,E$; on $A\,C$ as a diameter describe a circle cutting the given circle in B and D. Draw $A\,B$ and $A\,D$, and each will be tangent to the given circle through the given point. For drawing the radii $C\,B$, $C\,D$, the angles B, D, are each right angles (III. 25); therefore $A\,B$ and $A\,D$ are tangents to the given circle.

PROBLEM XV.

19. *Upon a given straight line to describe a segment of a circle which shall contain a given angle.*



Let $A\,B$ be the given straight line.

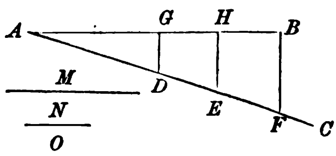
At B make the angle $A\,B\,D$ equal to the given angle (5). Draw $B\,C$ perpendicular to $D\,B$; bisect $A\,B$ in E, and from E draw $E\,C$ perpendicular to $A\,B$. From C, the point of intersection of $B\,C$ and $E\,C$, with a radius equal to $C\,B$, describe a circle $A\,G\,B\,F$; $B\,F\,A$ is the segment required. For $A\,B$ is a chord (I. 94). And as $B\,D$ is perpendicular to the radius $C\,B$ at B, it is a tangent to the circle; hence the angle $A\,B\,D$ is measured by half the arc $A\,G\,B$ (III. 29); and any angle $B\,F\,A$ inscribed in the segment $B\,F\,A$ is also measured by half the arc $A\,G\,B$ (III. 23), and is therefore equal to the angle $A\,B\,D$ or the given angle.

20. *Scholium.* The required segment can also be found by drawing AC and BC so as to make the angles CAB and ABC each equal to the complement of the given angle; and then C will be the centre and CA the radius of the required segment (III. 23).

21. *Corollary.* If the given angle is a right angle, the required segment would be a semicircle described on the given line as a diameter.

PROBLEM XVI.

22. *To divide a given line into parts proportional to given lines.*



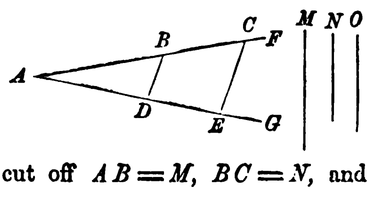
Let it be required to divide AB into parts proportional to M, N, O.

Draw at any angle with AB an indefinite line AC. From A cut off AD, DE, EF equal respectively to M, N, O. Join B to F, and through D and E draw lines parallel to BF. These parallels divide the line as required (II. 50).

23. *Corollary.* By taking M, N, O equal, the given line can be divided into equal parts.

PROBLEM XVII.

24. *To find a fourth proportional to three given lines.*



Let it be required to find a fourth proportional to M, N, O.

Draw at any angle with each other the indefinite lines AF, AG. From AF cut off $AB = M$, $BC = N$, and

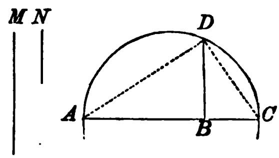
from AG cut off $AD = O$. Join BD and through C draw CE parallel to BD; then DE is the required fourth proportional (II. 50).

25. *Corollary.* By taking AB equal to M, and AD and BC each equal to N, a third proportional can be found to M and N.

PROBLEM XVIII.

26. *To find a mean proportional between two given lines.*

Let it be required to find a mean proportional between M and N.



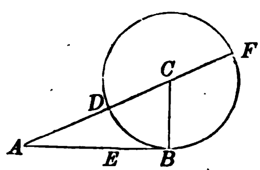
From an indefinite line cut off $AB = M$, $BC = N$; on AC as a diameter describe a semicircle, and at B draw BD perpendicular to AC. BD is the mean proportional required. Join AD, DC. (III. 25; II. 65.)

27. *Definition.* When a line is divided so that one segment is a mean proportional between the whole line and the other segment, it is said to be divided *in extreme and mean ratio.*

PROBLEM XIX.

28. *To divide a given line in extreme and mean ratio.*

Let it be required to divide AB in extreme and mean ratio.



At B draw the perpendicular $BC = \frac{1}{2} AB$; join AC; cut off $CD = CB$, $AE = AD$, and AB is divided at E in extreme and mean ratio.

For, describe a circle with the centre C and radius CB and produce AC to meet the circumference in F; then AF is a secant and AB a tangent of the circle DFB, and therefore (III. 86)

$$AF : AB = AB : AD$$

and (II. 18)

$$AF - AB : AB = AB - AD : AD$$

But $$AB = 2\,CB = DF$$

therefore $$AF - AB = AF - DF = AD = AE$$

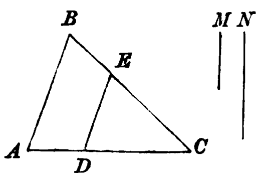
and the proportion becomes

$$AE : AB = EB : AE$$

or (II. 16) $$AB : AE = AE : EB$$

PROBLEM XX.

29. *Through a given point within the sides of a given angle to draw a line so that the segments included between the point and the sides of the angle may be in a given ratio.*



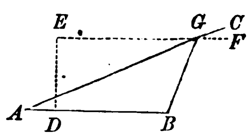
Let it be required to draw through the point D within the angle B a line so that $AD : DC = M : N$.

Draw DE parallel to AB.

Find EC a fourth proportional to M, N, and BE (24); join C to D, and produce CD to A, and AC is the line required (II. 50).

PROBLEM XXI.

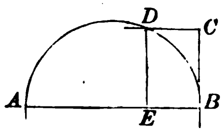
30. *The base, an adjacent angle, and the altitude of a triangle given, to construct the triangle.*



At A of the base AB draw an indefinite line AC making the angle A equal to the given angle; at any point in AB, as D, draw the perpendicular DE equal to the given altitude; through E draw EF parallel to AB cutting AC in G; join GB, and AGB is the triangle required.

PROBLEM XXII.

31. *To construct a parallelogram, having the sum of its base and altitude given, which shall be equivalent to a given square.*

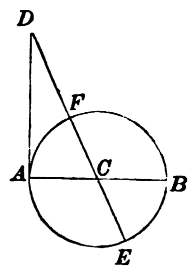


On $A\,B$, the given sum, as a diameter, describe a semicircumference. At any point, as B, in $A\,B$ draw the perpendicular $B\,C$ equal to a side of the given square; through C draw $C\,D$ parallel to $A\,B$, cutting the circumference in D; draw $D\,E$ perpendicular to $A\,B$. $A\,E$, $E\,B$ are one the base and the other the altitude of the parallelogram required (26).

32. *Scholium.* If the side of the square is greater than half the sum of the base and altitude, the construction is impossible.

PROBLEM XXIII.

33. *To construct a parallelogram having the difference between its base and altitude given, which shall be equivalent to a given square.*



On $A\ B$ the given difference, as a diameter, describe a circumference. At A draw the perpendicular $A\ D$ equal to a side of the given square; join D with the centre C, and produce $D\ C$ to E. $D\ F$, $D\ E$ are, one the base, and the other the altitude of the parallelogram required (III. 86).

PROBLEM XXIV.

34. *To construct a square equivalent to a given parallelogram.*

Find a mean proportional between the altitude and base of the given parallelogram (26), and it will be a side of the required square.

PROBLEM XXV.

35. *To construct a square equivalent to a given triangle.*

Find a mean proportional between the base and half the altitude (26), and it will be a side of the required square.

PROBLEM XXVI.

36. *To construct a square equivalent to a given circle.*

Find a mean proportional between the radius and the semicircumference, and it will be a side of the required square.

PROBLEM XXVII.

37. *To construct a square equivalent to the sum of two given squares.*

Construct a right triangle (9) with the sides adjacent to the right angle equal respectively to the sides of the given squares; the hypothenuse will be a side of the required square (II. 66).

38. *Scholium.* By continuing the same process we can find a square equivalent to the sum of any number of given squares.

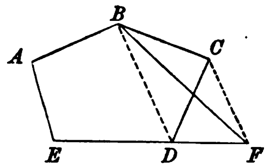
PROBLEM XXVIII.

39. *To construct a square equivalent to the difference of two given squares.*

Construct a right triangle (11), taking as the hypothenuse a side of the greater square, and for one of the sides adjacent to the right angle a side of the other square; the third side of the triangle will be a side of the required square (II. 67).

PROBLEM XXIX.

40. *To construct a triangle equivalent to a given polygon.*



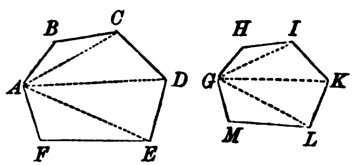
Let $A\ D$ be the polygon.

Draw $B\ D$ cutting off the triangle $B\ C\ D$; through C draw $C\ F$ parallel to $B\ D$ meeting $E\ D$ produced in F; join $B\ F$, and a polygon $A\ B\ F\ E$ is formed with one side less than the given polygon and equivalent to it. For the triangles $B\ C\ D$ and $B\ F\ D$, having the same base $B\ D$, and the same altitude, are equivalent; adding to each the common part $A\ B\ D\ E$, we have $A\ B\ C\ D\ E$ equivalent to $A\ B\ F\ E$. In like manner a polygon with one side less can be found equivalent to $A\ B\ F\ E$, and by continuing the process the sides may be reduced to three, and a triangle obtained equivalent to the given polygon.

41. *Scholium.* Since by (35) a square can be found equivalent to a given triangle, by (40) and (35) a square can be found equivalent to any polygon.

PROBLEM XXX.

42. *On a given line to construct a polygon similar to a given polygon.*



Let $A D$ be the given polygon and $M L$ the given line.

Draw the diagonals $A E$, $A D$, $A C$. At M and L make the angles $G M L$ and $G L M$ equal respectively to $A F E$ and $A E F$, and a triangle $G L M$ will be formed similar to $A E F$. In like manner on $G L$ construct a triangle similar to $A D E$; on $G K$ one similar to $A C D$; on $G I$ one similar to $A B C$; and the polygons $A D$,

KG, being composed of the same number of similar triangles similarly situated, are similar (II. 78).

PROBLEM XXXI.

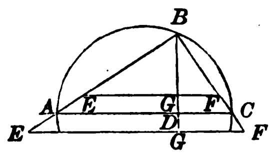
43. *Two similar polygons being given, to construct a similar polygon equivalent to their sum, or to their difference.*

Find a line whose square shall be equivalent to the sum (37), or to the difference (39), of the squares of any two homologous sides of the given polygons, and this will be the homologous side of the required polygon (II. 79). On this line construct (42) a polygon similar to the given polygons.

PROBLEM XXXII.

44. *To construct a square which shall be to a given square in a given ratio.*

On any line AC, as a diameter, describe a semicircumference ABC; divide the line AC at the point D so that $AD : DC$ in the given ratio. Perpendicular to AC draw DB meeting the circumference at B; join BA, BC, and on BC, produced if necessary, take $BF =$ a side of the given square. Through F draw EF parallel to AC, meeting BA in E, and BE is a side of the required square.



For as B is a right angle (III. 25), we have (II. 127)

$$BE^2 : BF^2 = EG : GF$$

But as EF is parallel to AC, we have (II. 101)

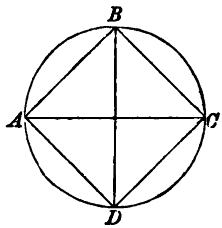
$$EG : GF = AD : DC$$

therefore (II. 11)

$$BE^2 : BF^2 = AD : DC$$

PROBLEM XXXIII.

45. *To inscribe a square in a given circle.*

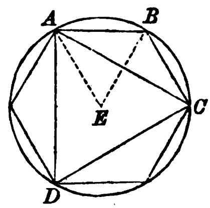


Draw two diameters AC, BD at right angles to each other, and join AB, BC, CD, DA; $ABCD$ is the required square (III. 25; III. 12).

46. *Corollary.* By bisecting the arcs AB, BC, CD, DA, and drawing the chords of these smaller arcs, a regular octagon will be inscribed in the circle. By continuing this bisection regular polygons can be inscribed having the number of their sides 16, 32, 64, and so on.

PROBLEM XXXIV.

47. *To inscribe a regular hexagon in a given circle.*



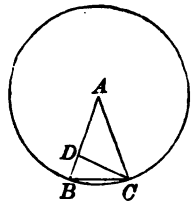
Take AB equal to the radius of the given circle, and it will be a side of the hexagon required (III. 51).

48. *Corollary.* By drawing AC, CD, DA an equilateral triangle will be inscribed in the circle. By bisecting the arcs AB, BC, &c., and continuing this bisection as in (46), and drawing the chords of these smaller arcs, regular polygons can be inscribed having the number of their sides 12, 24, 48, 96, and so on.

PROBLEM XXXV.

49. *To inscribe a regular decagon in a given circle.*

Divide the radius AB in extreme and mean ratio at the point D (28), and take $BC = AD$, the greater segment, and it will be the side of the required decagon.



Draw AC, CD. The triangles ACB, DCB are similar (II. 60); for they have the angle B common, and by construction

$$AB : AD = AD : DB$$

but $$AD = BC$$

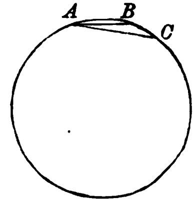
therefore $$AB : BC = BC : BD$$

Therefore, as ACB is isosceles, DCB is also isosceles, and $CD = CB$; therefore also $CD = DA$, and ACD is an isosceles triangle, and the angle $A = ACD$. But the exterior angle $BDC = A + ACD =$ twice the angle A. Therefore, as $B = BDC$, $B =$ twice the angle A. But $B = ACB$; therefore the sum of the three angles A, B, and ACB is equal to five times the angle A; or the angle A is one fifth of two right angles, or one tenth of four right angles; therefore the arc BC is one tenth of the circumference, and the chord BC a side of a regular decagon inscribed in the circle.

50. *Corollary.* By drawing chords joining the alternate angles a regular pentagon will be inscribed. By proceeding as in (46) regular polygons can be inscribed having the number of their sides 20, 40, 80, and so on.

PROBLEM XXXVI.

51. *To inscribe a regular polygon of fifteen sides in a given circle.*



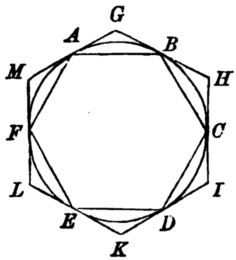
Find by (47) the arc AC equal to a sixth of the circumference, and by (49) the arc AB equal to a tenth of the circumference, and the chord BC will be a side of the polygon required.

For $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$

52. *Corollary.* Proceeding as in (46) regular polygons can be inscribed having the number of their sides 30, 60, and so on.

PROBLEM XXXVII.

53. *To circumscribe about a given circle a polygon similar to a given inscribed regular polygon.*



Let $A\,D$ be the given inscribed regular polygon. A similar polygon can be circumscribed about the circle $A\,C\,E$ in the same manner as shown in (III. 33). Or, through the points A, B, C, D, E, F, draw tangents to the circumference. These tangents intersecting will form the polygon required.

For as $FM = MA = A\,G = G\,B = B\,H$, &c. (III. 32), therefore $M\,G = G\,H = H\,I$, &c., and the circumscribed polygon is equilateral; and as the angle $M = G = H$, &c. (III. 32), therefore the circumscribed polygon is regular (II. 80); as it has the same number of sides as the inscribed polygon, it is similar to it (II. 81); and as its sides are tangents, it is circumscribed about the circle.

54. *Corollary.* As (45 – 52) regular polygons can be inscribed having the number of their sides 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 64, 80, 96, and so on, regular polygons having the number of their sides represented by these numbers can also be circumscribed about a given circle.

EXERCISES.

55. From two given points to draw two equal lines meeting in a given straight line. (I. 94.)

56. Through a given point to draw a line at equal distances from two other given points.

57. From a given point out of a straight line to draw a line making a given angle with that line. (I. 54.)

58. From two given points on the same side of a given line to draw two lines meeting in the first line and making equal angles with it.

59. From a given point to draw a line making equal angles with the sides of a given angle.

60. Through a given point to draw a line so that the parts of the line intercepted between this point and perpendiculars from two other given points shall be equal.

If the three points are in a straight line, the parts equal what?

61. From a point without two given lines to draw a line such that the part between the two lines shall be equal to the part between the given point and the nearer line.

When is the Problem impossible?

62. To trisect a right angle.

63. On a given base to construct an isosceles triangle having each of the angles at the base double the third angle.

64. To construct an isosceles triangle when there are given

1st. The base and opposite angle.

2d. The base and an adjacent angle.

3d. A side and an opposite angle.

4th. A side and the angle opposite the base.

65. The base, opposite angle, and the altitude given, to construct the triangle. (III. 24.) (20.)

When is the Problem impossible?

66. The base, an angle at the base, and the sum of the sides given, to construct the triangle.

When is the Problem impossible?

67. The base, an angle at the base, and the difference of the sides given, to construct the triangle.

1st. When the given angle is adjacent to the shorter side.

2d. When the given angle is adjacent to the longer side.

When is the Problem impossible?

68. The base, the difference of the sides, and the difference of the angles at the base given, to construct the triangle.

69. The base, the angle at the vertex, and the sum of the sides given, to construct the triangle.

When is the Problem impossible?

70. The base, the angle at the vertex, and the difference of the sides given, to construct the triangle.

71. On a given base to construct a triangle equivalent to a given triangle.

72. With a given altitude to construct a triangle equivalent to a given triangle.

73. Two sides of a triangle and the perpendicular to one of them from the opposite vertex given, to construct the triangle.

74. Two of the perpendiculars from the vertices to the opposite sides and a side given, to construct the triangle.

1st. When one of the perpendiculars falls on the given side.

2d. When neither of the perpendiculars falls on the given side.

75. An angle and two of the perpendiculars from the vertices to the opposite sides given, to construct the triangle.

1st. When one of the perpendiculars falls from the vertex of the given angle.

2d. When neither of the perpendiculars falls from the vertex of the given angle.

76. An angle and the segments of the opposite side made by a perpendicular from the vertex given, to construct the triangle.

77. Given an angle, the opposite side, and the line from the given vertex to the middle of the given side, to construct the triangle.

When is the Problem impossible?

78. An angle, a perpendicular from another angle to the opposite side, and the radius of the circumscribed circle given, to construct the triangle.

When is the Problem impossible?

79. To divide a triangle into two parts in a given ratio,

1st. By a line drawn from a given point in one of its sides.

2d. By a line parallel to the base.

80. To trisect a triangle by straight lines drawn from a point within to the vertices.

81. Parallel to the base of a triangle to draw a line equal to the sum of the lower segments of the two sides.

82. Parallel to the base of a triangle to draw a line equal to the difference of the lower segments of the two sides.

83. To inscribe in a given triangle a quadrilateral similar to a given quadrilateral.

84. To divide a given line so that the sum of the squares of the parts shall be equivalent to a given square.

85. To construct a parallelogram when there are given,

1st. Two adjacent sides and a diagonal.
2d. A side and two diagonals.
3d. The two diagonals and the angle between them.
4th. The perimeter, a side, and an angle.

86. To construct a square when the diagonal is given.

87. To construct a parallelogram equivalent to a given triangle and having a given angle.

88. To draw a quadrilateral, the order and magnitude of all the sides and one angle given.

Show that sometimes there may be two different polygons satisfying the conditions.

89. To draw a quadrilateral, the order and magnitude of three sides and two angles given.

1st. The given angles included by the given sides.
2d. The two angles adjacent, and one adjacent to the unknown side.
3d. The two angles being opposite each other.
4th. The two angles being both adjacent to the unknown side.

In any of these cases can more than one quadrilateral be drawn?

90. To draw a quadrilateral, the order and magnitude of two sides and three angles given.

1st. The given sides being adjacent.
2d. The given sides not being adjacent.

91. In a given circle to inscribe a triangle similar to a given triangle.

92. Through a given point to draw to a given circle a secant such that the part within the circle may be equal to a given line.

93. With a given radius to draw a circumference,

1st. Through two given points.

2d. Through a given point and tangent to a given line.

3d. Through a given point and tangent to a given circumference.

4th. Tangent to two given straight lines.

5th. Tangent to a given straight line and to a given circumference.

6th. Tangent to two given circumferences.

State in each of these cases how many circles can be drawn, and when the construction is impossible.

94. To draw a circumference,

1st. Through two given points and with its centre in a given line.

2d. Through a given point and tangent to a given line at a given point.

3d. Tangent to a given line at a given point, and also tangent to a second given line.

4th. Tangent to three given lines.

5th. Through two given points and tangent to a given line.

6th. Through a given point and tangent to two given lines.

95. To draw a tangent to two circumferences.

There can be drawn,

1st. When the circles are external to each other, four tangents.

2d. When the circles touch externally, three.

3d. When the circles cut, two.

4th. When the circles touch internally, one.

5th. When one circle is within the other, none.

BOOK VI.

GEOMETRY OF SPACE.

PLANES AND THEIR ANGLES.

DEFINITION.

1. Geometry of Space, or geometry of three dimensions, treats of figures whose elements are not all in the same plane.

For the definition of a plane see I. 11.

THEOREM I.

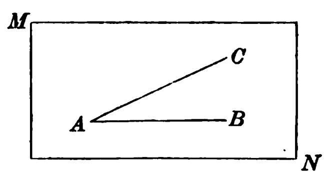
2. *A plane is determined,*

1st. *By a straight line and a point without that line;*

2d. *By three points not in the same straight line;*

3d. *By two intersecting straight lines.*

1st. Let the plane $M N$, passing through the line $A B$, turn upon this line as an axis until it contains the point C; the position of the plane is evidently determined; for if it is turned in either direction it will no longer contain the point C.



2d. If three points, A, B, C, not in the same straight line, are given, any two of them, as A and B, may be joined by a straight line; then this is the same as the 1st case.

3d. If two intersecting lines $A B$, $A C$, are given, any point, C, out of the line $A B$ can be taken in the line $A C$; then the plane passing through the line $A B$ and the point C contains the two lines $A B$ and $A C$, and is determined by them.

3. *Cor.* 1. The intersection of two planes is a straight line; for the intersection cannot contain three points not in the same straight line, since only one plane can contain three such points.

4. *Cor.* 2. Through a straight line an infinite number of planes can pass. For the plane MN, revolving on AB as an axis, occupies an infinite number of positions.

DEFINITIONS.

5. A straight line is *perpendicular to a plane* when it is perpendicular to every straight line of the plane which it meets.

Conversely, the plane, in this case, is perpendicular to the line.

The *foot* of the perpendicular is the point in which it meets the plane.

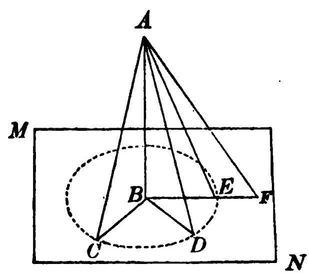
THEOREM II.

6. *There can be but one perpendicular from a point to a plane.*

If there could be two, they would be in the same plane (2); and the intersection of this plane with the given plane would be a straight line (3), and then there would be two perpendiculars from a point to a straight line, (the three lines in the same plane,) which is impossible (I. 57).

THEOREM III.

7. *Oblique lines from a point to a plane equally distant from the perpendicular are equal; and of two oblique lines unequally distant from the perpendicular, the more remote is the greater.*



Let AC, AD be oblique lines drawn to the plane MN at equal distances from the perpendicular AB:

1st. $AC = AD$; for the triangles ABC, ABD are equal (I. 80).

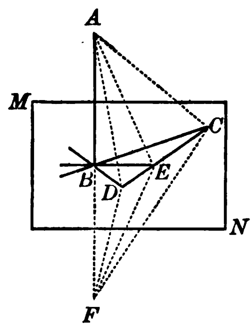
2d. Let AF be more remote. From BF cut off $BE = BD$ and draw AE; then $AF > AE$ (I. 90); and $AE = AD = AC$; therefore $AF > AD$ or AC.

8. *Cor.* 1. *Conversely*, equal oblique lines from a point to a plane are equally distant from the perpendicular; therefore they meet the plane in the circumference of a circle whose centre is the foot of the perpendicular. Of two unequal lines the greater is more remote from the perpendicular.

9. *Cor.* 2. The perpendicular is the shortest distance from a point to a plane.

THEOREM IV.

10. *A line perpendicular to each of two lines at their point of intersection is perpendicular to the plane of these lines.*



Let AB be perpendicular to BC, BD at their point of intersection B; then AB is perpendicular to the plane MN, in which the lines BC, BD are.

Let BE be any other line through B in the plane MN. Draw a line intersecting BC, BE, BD in CED; produce AB so as to make $BF = AB$; join AC, AE, AD, FC, FE, FD.

As BC and BD are perpendicular to AF at its middle point, the triangles ACD, DEF have (I. 94) $AC = CF$ and $AD = DF$; and CD is common; therefore (I. 88) the triangle $ACD = DCF$, and the equal angles at the base DC are adjacent; hence lines drawn from the corresponding vertices A and F to corresponding points of their bases must be equal; that is, $AE = EF$. Hence E must be a point in a perpendicular passing through B the middle of AF (converse of I. 94), that is, AB is perpendicular to BE. Therefore AB is perpendicular to the plane MN (5).

11. *Corollary.* Hence to pass a plane through any point D perpendicular to a given line AF, draw a perpendicular from the point D to the line AF, and at B, the point of intersection with the given line, and not in the plane of the given line

$A F$ and the perpendicular $D B$, draw a second perpendicular $B C$ to the given line $A F$; these two perpendiculars $B C$, $B D$ determine the position of the plane (2). But one such plane can be drawn; for through a given point there can be but one perpendicular to a line in the same plane with that line (I. 39).

THEOREM V.

12. *Parallel lines are equally inclined to the same plane.*

For parallel lines have the same direction (I. 50); they must therefore have the same difference of direction from the same plane, that is, be equally inclined to it.

13. *Scholium* 1. The converse of this, namely, that lines equally inclined to the same plane are parallel, is not necessarily true. For example, $A C$ and $A D$, in the Fig. in Theorem III., equally inclined to the same plane $M N$, are not parallel.

14. *Scholium* 2. It is evident, however, that lines perpendicular to the same plane are parallel; as in this case they must have the same direction with each other.

DEFINITIONS.

15. **The Normal to a Plane** at any point is the perpendicular to the plane at that point.

A plane has an infinite number of normals, but all have the same direction (14).

16. The **Aspect of a Plane** is the direction of its normals. As the normals all have the same direction, a plane has the same aspect throughout. If we consider both faces of the plane, as a straight line has two directions (I. 7), so a plane has two aspects exactly opposite to each other.

17. *A line and a plane are parallel* when the line is perpendicular to a normal of the plane.

18. *Corollary.* A line parallel to a plane can never meet the plane, however far produced. For if they coincide at any point, both being perpendicular to the normal, the line must lie wholly in the plane (5; I. 39).

Conversely, a line that never meets a plane, however far produced, must be parallel to the plane. For, if it never meets the plane, it cannot be approaching the plane in either direction, that is, it must be perpendicular to the normal, and therefore parallel to the plane.

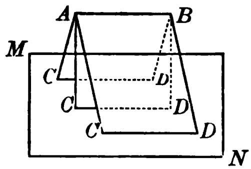
19. Parallel Planes are such as have the same aspect.

20. *Corollary.* Parallel planes can never meet. For, since parallel planes have the same aspect, if they coincided at any point, they would coincide throughout, and form one and the same plane.

Conversely, planes that never meet, however far produced, are parallel. For, if they never meet, they cannot be approaching in any direction, that is, they cannot have different aspects, but must have the same aspect.

THEOREM VI.

21. *A line parallel to a plane is also parallel to the intersection with this plane of any plane passing through the line.*

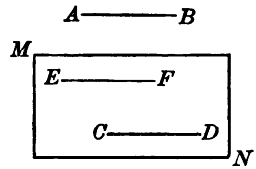


Let $A\,B$, be parallel to the plane $M\,N$, and let $C\,D$ be the intersection with $M\,N$ of any plane $A\,D$ passing through $A\,B$; $A\,B$ is parallel to $C\,D$.

For $A\,B$, being parallel to the plane $M\,N$, can never meet $M\,N$ (18), and therefore can never meet $C\,D$ which is in $M\,N$; hence, as $A\,B$ and $C\,D$ are in the same plane $A\,D$ and never meet, they are parallel (I. 51).

THEOREM VII.

22. *A plane passing through one of two parallel lines is parallel to the other.*



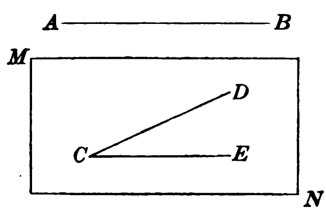
Let the plane $M N$ pass through the line $C D$, one of the two parallel lines $A B$, $C D$; then the plane $M N$ and the line $A B$ are parallel.

For, if $A B$ is not parallel to the plane $M N$, it will, if produced, meet the plane in some point X. Through X draw a line parallel to $C D$, and let $E F$ be a part of this line. $A B$ and $E F$, being both parallel to $C D$, are parallel to each other (I. 52); but $A B$ and $E F$ are supposed to meet at X, which is absurd (I. 51). Therefore $A B$ cannot meet the plane $M N$; that is, $A B$ is parallel to the plane $M N$ (18).

23. *Scholium.* If the plane $M N$ revolves on $C D$, there will be one position in which it will include in its surface both parallels $A B$ and $C D$.

THEOREM VIII.

24. *Through any line there can be passed a plane parallel to a given line.*



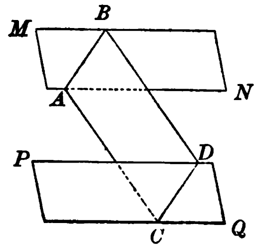
Through $C D$ let it be required to pass a plane parallel to $A B$.

Through C draw $C E$ parallel to $A B$, and pass a plane, $M N$, through $C D$, $C E$ (2). This plane is parallel to $A B$ (22).

25. *Corollary.* There can be but one plane passing through $C D$ which will be parallel to $A B$, unless $A B$ is parallel to $C D$; in this case there can be an infinite number (4; 22).

THEOREM IX.

26. *The intersections of two parallel planes with a third plane are parallel.*



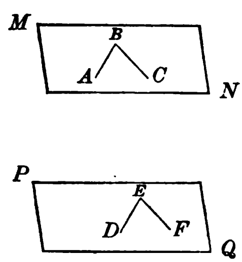
Let $A B$ and $C D$ be the intersections of the plane $A D$ with the parallel planes $M N$ and $P Q$; then $A B$ and $C D$ are parallel.

For the lines $A B$ and $C D$ cannot meet though produced indefinitely, since the planes $M N$ and $P Q$ in which they are cannot meet; and they are in the same plane $A D$; therefore they are parallel.

27. *Corollary.* Parallels intercepted between parallel planes are equal. For the opposite sides of the quadrilateral $A D$ being parallel, the figure is a parallelogram; therefore $A C = B D$.

THEOREM X.

28. *If two angles not in the same plane have their sides parallel and similarly situated, the angles are equal and their planes parallel.*



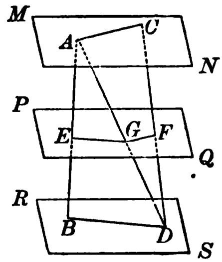
Let $A B C$ and $D E F$ be two angles in the planes $M N$ and $P Q$, having their sides $A B$, $B C$ respectively parallel to $D E$, $E F$, and similarly situated; then

1st. Since $B A$ has the same direction as $E D$, and $B C$ the same as $E F$, the difference of direction of $B A$ and $B C$ must be the same as the difference of direction of $E D$ and $E F$; that is, angle $B =$ angle E.

2d. The planes of these angles are parallel. For, since two intersecting lines determine a plane (2), the plane of the lines $A B$ and $B C$ must be parallel to the plane of the lines $D E$ and $E F$, as $A B$ and $B C$ are respectively parallel to $D E$ and $E F$.

THEOREM XI.

29. *If two straight lines are cut by parallel planes, they are divided proportionally.*



Let $A B$ and $C D$ be cut by the parallel planes $M N$, $P Q$, and $R S$, in the points $A E B$, and $C F D$; then

$$A E : E B = C F : F D$$

For, drawing $A D$ meeting the plane $P Q$ in G, the plane of the lines $A B$ and $A D$ cuts the parallel planes $P Q$ and $R S$ in $E G$ and $B D$; therefore $E G$ and $B D$ are parallel (26), and we have (II. 50)

$$A E : E B = A G : G D$$

The plane of the lines $A D$ and $C D$ cuts the parallel planes $M N$ and $P Q$ in $A C$ and $G F$; therefore $A C$ is parallel to $G F$; and we have

$$A G : G D = C F : F D$$

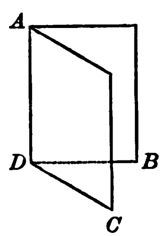
Hence we have (II. 11)

$$A E : E B = C F : F D$$

DIEDRAL ANGLES.

DEFINITIONS.

30. A **Diedral Angle** is the difference in aspect of two planes. Two planes, not parallel, necessarily intersect.



Thus the planes $A B$, $A C$, intersecting in the line $A D$, form a diedral angle.

The intersecting planes $A B$, $A C$, are called the *faces*, and the line of intersection, $A D$, the *edge* of the diedral angle.

31. The angle formed by two lines, one in

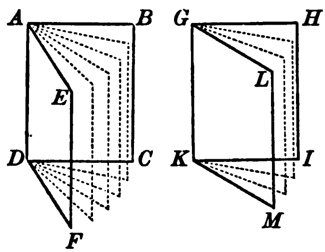
each plane, drawn from a common point in the line of intersection and each perpendicular to that line, is called *the plane angle of the diedral angle.*

Thus, if $B\,D$, $D\,C$ are each perpendicular to $A\,D$, $B\,D\,C$ is the plane angle of the diedral angle formed by the planes $A\,B$, $A\,C$.

32. *Corollary.* It is evident from (28) that all the plane angles of a diedral angle are equal.

THEOREM XII.

33. *Diedral angles are to one another as their plane angles.*



Let $B\,A\,E$, $H\,G\,L$ be the plane angles of the diedral angles formed respectively by the planes $A\,C$, $A\,F$, and the planes $G\,I$, $G\,M$.

Let the plane angles be as $5:3$; then the diedral angles are as $5:3$.

Suppose the plane angles have a common measure which is contained, for example, in $B\,A\,E$ five times and in $H\,G\,L$ three times; through the lines of division of the angle $B\,A\,E$ and the edge $A\,D$ let planes be passed, and also through the lines of division of $H\,G\,L$ and the edge $G\,K$; the small diedral angles thus formed are equal to one another; for if applied to one another their faces would coincide. The diedral angle formed by the faces $A\,C$, $A\,F$ contains five of the small equal diedral angles; and the diedral angle formed by the faces $G\,I$, $G\,M$ contains three; therefore the diedral angles are as $5:3$, that is, as the plane angles $B\,A\,E$, $H\,G\,L$.

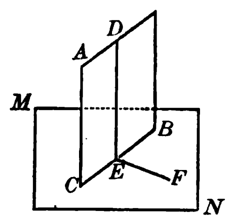
The proof is extended to the case in which the plane angles $B\,A\,E$, $H\,G\,L$ have no common measure, by the method shown in (II. 35).

34. *Corollary.* As diedral angles vary as their plane angles, the plane angle is taken as the measure of the diedral angle.

35. *Definition.* When two planes cut one another so as to make the adjacent diedral angles equal, each of these angles is a *right diedral angle,* and the planes are perpendicular to each other.

THEOREM XIII.

36. *If two planes are perpendicular to each other, any line in one, perpendicular to the common intersection of the planes, is perpendicular to the other plane.*



Let the planes MN and AB be perpendicular to each other, and DE in the plane AB be perpendicular to BC the intersection of the planes; then DE is perpendicular to the plane MN.

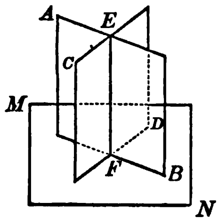
Through E draw EF in the plane MN perpendicular to BC; as the planes are perpendicular to each other the plane angle DEF, which is the measure of the right diedral angle (34), is a right angle; therefore DE being perpendicular to the two lines BC, EF is perpendicular to the plane of these lines MN (10).

37. *Corollary.* If two planes are perpendicular to each other, a straight line, perpendicular to one of the planes, passing through any point of the other plane, must lie wholly in the other plane.

THEOREM XIV.

38. *If two planes are perpendicular to a third plane, their intersection is perpendicular to this third plane.*

Let the planes $A B$, $C D$, intersecting in the line $E F$, be perpendicular to the plane $M N$; then $E F$ is perpendicular to the plane $M N$.

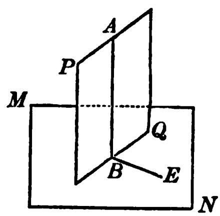


For (37) a line drawn through F perpendicular to the plane $M N$ must lie in both $A B$ and $C D$, that is, must be their intersection.

THEOREM XV.

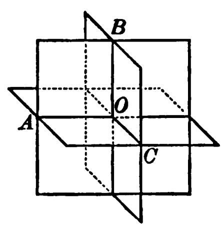
39. *If a line is perpendicular to a plane every plane passing through this line is perpendicular to that plane.*

Let $A B$ be perpendicular to the plane $M N$, then any plane $P Q$ passing through $A B$ is perpendicular to $M N$.



In the plane $M N$ draw $B E$ perpendicular to the intersection $B Q$. As $A B$ is perpendicular to $M N$ the plane angle $A B E$ is a right angle (5); therefore the plane $P Q$ is perpendicular to $M N$ (34).

40. *Corollary.* If three lines $A O$, $B O$, $C O$, are perpendicular to each other at a common point O, each line is perpendicular to the plane of the other two lines (10), and the three planes are perpendicalar to each other.

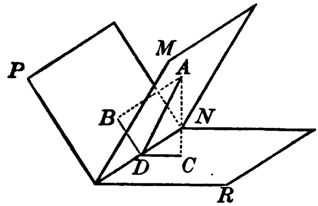


THEOREM XVI.

41. *Every point in a plane bisecting a diedral angle is equally distant from the faces of the angle.*

Let the plane $M N$ bisect the diedral angle whose faces are

$P N$, $N R$; then any point A in the plane $M N$ is equally distant from the planes $P N$, $N R$; that is, if we draw $A B$, $A C$ perpendicular respectively to the planes, $P N$, $N R$, then

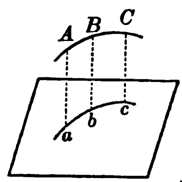


$$A B = A C$$

Through $A B$, $A C$ pass a plane whose intersections with the planes $M N$, $P N$, $N R$ are $A D$, $B D$, $D C$. The plane $A B C$ is perpendicular to each of the planes $P N$, $N R$ (39), and therefore perpendicular to their intersection $D N$ (38), and $D B$, $D A$, $D C$ are each perpendicular to $D N$ (5). Hence the angles $A D B$, $A D C$ are the measures (34) of the diedral angles made by the plane $M N$ with the planes $P N$, $N R$, respectively; as the diedral angles are equal, the plane angles $A D B$, $A D C$ are equal; and the angles $A B D$, $A D C$ are each right angles (5). And, therefore, as the side $A D$ is common, the triangles $B A D$, $D A C$ are equal (I. 81); hence $A B = A C$.

DEFINITIONS.

42. The *projection of a point upon a plane* is the point where the perpendicular, drawn from the point to the plane, meets the plane.



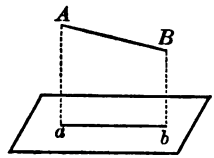
Thus, if $A\,a$ is perpendicular to the plane $M N$, a is the projection of the point A on the plane $M N$.

The line $A\,a$ is called the *projecting line* of the point A.

43. The *projection of a line upon a plane* is the line formed by joining the projections of all the points of the given line upon the plane. Thus if through all the points of the line $A B C$ perpendiculars are drawn to the plane $M N$, and through these points a line $a\,b\,c$ is drawn, $a\,b\,c$ will be the projection of $A B C$ on the plane $M N$.

THEOREM XVII.

44. *The projection of a straight line upon a plane is a straight line.*



Let $A B$ be the given line and $M N$ the given plane. Let $A a$ be the projecting line of the point A; the plane passing through $A B$ and $A a$ must contain all the projecting lines of the points of $A B$ (37); that is, these projecting lines all meet the plane $M N$ in the line of intersection of the planes $A b$, $M N$; hence the line of projection $a b$ is a straight line (3).

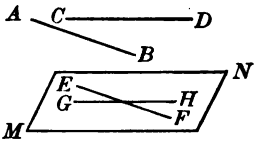
45. *Cor.* 1. It is evident that the projection of any line upon a plane parallel to this line is a line parallel to (22), and exactly like, the given line.

46. *Cor.* 2. It follows that through any line a plane can be passed perpendicular to a given plane. For the plane containing the projecting lines of the points of the given line is perpendicular to the given plane; and as the required plane must contain all these projecting lines there can be but one such plane (2); unless the given line is perpendicular to the given plane, in which case there can be an infinite number (39).

47. *Definition.* The plane including the projecting lines of a given line is called the *projecting plane* of the line. Thus $A b$ is the projecting plane of the line $A B$.

THEOREM XVIII.

48. *Through a given point there can be passed a plane parallel to any two given straight lines.*



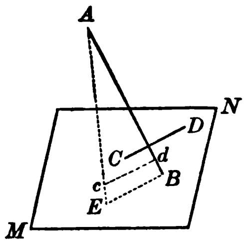
Let O be the given point, $A B$, and $C D$ the given lines.

Through O draw $E F$ and $G H$ parallel respectively to $A B$ and $C D$; through the lines $E F$ and $C D$ pass a plane $M N$ (2); the plane $M N$ is parallel to each of the lines $A B$ and $C D$ (22).

49. *Corollary.* The angle which the two lines $A B$, $C D$, not in the same plane, make with each other is equal to the angle which the two lines $E F$, $G H$, in the plane $M N$, and parallel respectively to $A B$, $C D$, make with each other; that is, it is equal to the angle which their projections upon a plane, parallel to both lines, make with each other.

THEOREM XIX.

50. *The angle which a line meeting a plane makes with any line in that plane is equal to the angle made by the first line and a line passing through its foot in the plane and parallel to the second line.*



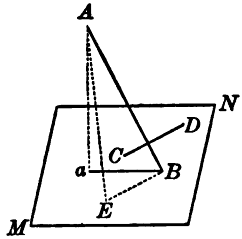
Let $A B$ be a line meeting the plane $M N$ in B, and $C D$ any line in the plane $M N$; then if $B E$ is drawn in the plane $M N$ parallel to $D C$, the angle $A B E$ is equal to the angle which $A B$ and $C D$ make with each other.

Through $A B$ and $B E$ pass a plane $A B E$; the plane $A B E$ is parallel to to $C D$ (22). Let $c d$ be the projection of $C D$ on the plane $A B E$; $c d$ is parallel to $C D$ (45), and therefore to $E B$ (I. 52). As $A B$ is its own projection in the plane $A B E$, the angle which $A B$ and $C D$ make with each other is equal to the angle $A d c$ (49); but $A d c = A B E$ (I. 54); hence the angle which $A B$ makes with $C D$ is equal to $A B E$.

51. *Corollary.* A line perpendicular to a plane is perpendicular to every line in that plane.

THEOREM XX.

52. *The acute angle which a straight line makes with its line of projection upon a plane is less than any angle which it makes with any other line in that plane.*



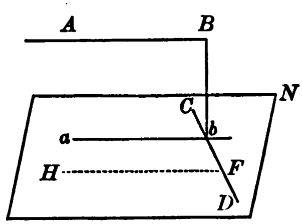
Let $a B$ be the projection of the straight line $A B$ upon the plane $M N$, the point B being the point of intersection of the given line and its projection; let $C D$ be any other straight line in the plane $M N$; through the point B, in the plane $M N$, draw $B E$ parallel to $D C$. The angle which $A B$ makes with $C D$ is equal to $A B E$ (50). Then the angle $A B a < A B E$.

Take $B E = B a$; join $A E$. The triangle $A B a$ has the two sides $A B$, $B a$ equal respectively to the sides $A B$, $B E$ of the triangle $A B E$; but $A a < A E$ (9). Therefore (I. 102) the angle $A B a < A B E$, that is, less than the angle which $A B$ makes with $C D$.

53. *Definition.* The angle which a straight line makes with its line of projection upon a plane is called *the inclination of the line to the plane.*

THEOREM XXI.

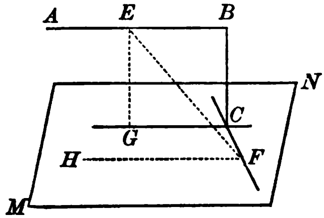
54. *A common perpendicular can be drawn to two given straight lines not in the same plane.*



Let $A B$, $C D$ be the given lines.

Through $C D$ pass a plane $M N$ parallel to $A B$ (24); let $a b$ be the projection of $A B$ on $M N$. Then $a b$ is parallel to $A B$ (21), and hence, as $A B$ and

$C D$ are not in the same plane, cannot be parallel to $C D$; therefore $a b$ will meet $C D$; let it meet it in b. At b draw $b B$ perpendicular to $a b$ in the projecting plane of the line $A B$. Then, as $A B$ is parallel to $a b$, $B b$ is perpendicular to $A B$; and as $B b$ is the projecting line of the point B on the plane $M N$, $B b$ is perpendicular to $M N$, and therefore to $C D$ (5).



55. *Cor.* 1. But one common perpendicular can be drawn to two lines not in the same plane. For if there could be another let it be $E F$; then, as $E F$ is perpendicular to $A B$ it is also perpendicular to $F H$, a line drawn in the plane $M N$ parallel to $A B$; and hence $E F$ being perpendicular to both $F C$ and $F H$, is perpendicular to their plane $M N$ (10); but $E G$, the projecting line of the point E upon the plane $M N$, is also perpendicular to the plane $M N$; hence there are two perpendiculars from the point E to the plane $M N$, which is impossible (6).

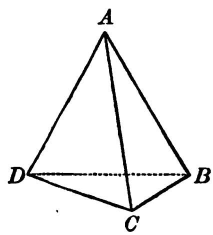
56. *Cor.* 2. The common perpendicular is the shortest distance between two lines not in the same plane. For any other distance $E F$ is greater than $E G$ (9), which is equal to $B b$.

POLYEDRAL ANGLES.

DEFINITIONS.

57. When three or more planes meet in a common point, they form a **Polyedral Angle**.

Thus, the planes $A B D$, $A B C$, $A C D$ meeting in the common point A form a polyedral angle at A.



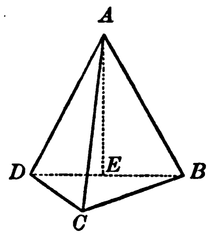
The point A is called the *vertex;* the intersections of the planes, $A B$, $A C$, $A D$, the *edges;* the parts of the planes within the edges, the *faces;* and the plane angles,

DAB, BAC, CAD, the *face angles* of the polyedral angle formed at A.

58. A **Triedral Angle** is a polyedral angle having only three faces.

THEOREM XXII.

59. *The sum of any two face angles of a triedral angle is greater than the third.*



Let A be a triedral angle whose face angles are DAB, BAC, and CAD, of which DAB is the greatest; we have only to prove $DAC + CAB > DAB$.

In the plane DAB draw AE making the angle $DAE = DAC$; through any point E of AE, in the plane DAB, draw the line DB; cut off $AC = AE$; join DC, CB.

The triangles DAE, DAC are equal (I. 80); hence $DE = DC$. In the triangle DBC

$$DC + CB > DB$$

and subtracting $\qquad DC = DE$

we have $\qquad CB > EB$

Then in the two triangles CAB, EAB the sides CA, AB are respectively equal to EA, AB, but the side $CB > EB$; hence (I. 102) the angle

$$CAB > EAB$$

adding $\qquad DAC = DAE$

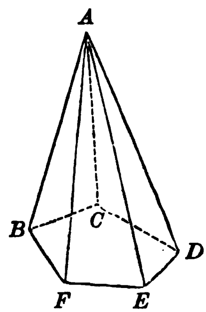
we have $\qquad DAC + CAB > DAB$

THEOREM XXIII.

60. *The sum of the face angles of a polyedral angle is less than four right angles.*

Let A be the vertex of the polyedral angle whose face angles are BAC, CAD, DAE, EAF, FAB; then

$$BAC + CAD + DAE + EAF + FAB < 4 \text{ right angles.}$$



Let the faces be cut by a plane forming the polygon $BCDEF$.

Let n represent the number of sides of the polygon BD, which is the same as the number of triangles, BAC, CAD, etc., whose common vertex is A; and let s represent the sum of the face angles of the polyedral angle, that is, the sum of all the angles at the vertices of the several triangles BAC, CAD, etc.

Now the sum of the angles at the bases of these triangles (I. 73) is equal to $2n - s$ right angles; and the sum of the angles of the polygon (I. 125) is equal to $2(n-2)$, or $2n - 4$, right angles. But in the triedral angles at B, C, D, etc., by (59)

we have $$ABC + ABF > FBC$$

$$ACB + ACD > BCD, \text{ etc.}$$

that is, $$2n - s > 2n - 4$$

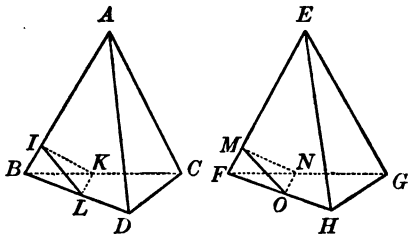
or, $$s < 4$$

that is, the sum of the face angles forming the polyedral angle A is less than four right angles.

61. *Scholium.* This demonstration assumes that the polyedral angle is convex, that is, that none of the faces produced would cut the polyedral angle. If the polyedral angle were not convex the sum of the face angles would be unlimited.

THEOREM XXIV.

62. *Triedral angles whose face angles are equal and similarly situated are equal.*



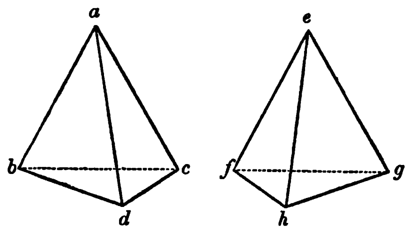
Let A be the vertex of the triedral angle whose face angles $B\,A\,C$, $C\,A\,D$, $D\,A\,B$ are respectively equal to the face angles $F\,E\,G$, $G\,E\,H$, $H\,E\,F$ of the triedral angle whose vertex is E; and let the faces in each be arranged in the same order; then the triedral angles are equal.

On the edges of the triedral angles cut off the six equal distances $A\,B$, $A\,C$, $A\,D$, $E\,F$, $E\,G$, $E\,H$; through the points B, C, D pass a plane (2) whose intersections with the faces of the triedral angle A are $B\,C$, $C\,D$, $D\,B$; also through the points F, G, H pass a plane whose intersections with the faces of the triedral angle E are $F\,G$, $G\,H$, $H\,F$. The isosceles triangle $B\,A\,C = F\,E\,G$ (I. 80), and hence $B\,C = F\,G$; in like manner $C\,D = G\,H$, and $B\,D = F\,H$; hence the triangles $B\,C\,D$, $F\,G\,H$, being mutually equilateral, are equal (I. 88), and the angle $D\,B\,C = H\,F\,G$. At any point I in $A\,B$, draw $I\,K$ in the face $B\,A\,C$, and $I\,L$ in the face $B\,A\,D$, perpendicular to $A\,B$; as $B\,A\,C$, $B\,A\,D$ are isosceles triangles, the angles $A\,B\,C$, $A\,B\,D$, are acute; hence $I\,K$, $I\,L$ will meet $B\,C$, $B\,D$ respectively; let K and L be the points of meeting; join $K\,L$. From $E\,F$ cut off $E\,M = A\,I$ and construct $M\,N\,O$ in the same manner as $I\,K\,L$ was constructed.

Now as $A\,B = E\,F$ and $A\,I = E\,M$, $I\,B = M\,F$; the angle $A\,B\,C = E\,F\,G$, and the right triangles $B\,I\,K$, $F\,M\,N$ are equal (I. 81) and $I\,K = M\,N$ and $B\,K = F\,N$. In like manner it can be proved that $I\,L = M\,O$ and $B\,L = F\,O$. Hence the

triangle $BKL = FNO$ (I. 80). Now the triangles IKL, MNO, being mutually equilateral are equal (I. 88), and the angle LIK, the measure (34) of the diedral angle whose edge is AB is equal to DMN, the measure of the diedral angle whose edge is EF. Now if the triedral angle A is placed on E so that the face BAC coincides with its equal FEG, as the diedral angles whose edges are AB, EF are equal; the face BAD will lie on its equal FEH, and AD on EH; hence DAC will coincide with HEG (2), and the triedral angle A with the triedral angle E.

63. *Scholium* 1. If the faces of the triedral angles are not *similarly* situated, as in the triedral angles whose vertices are a and e, the diedral angles whose edges are ab and eg can be proved equal in the same manner as in (61); and also the diedral angles whose edges are ac and ef; but the two figures cannot be made to coincide. In this case the two triedral angles are said to be *equal by symmetry*, or *symmetrically equal*, or simply *symmetrical*.



Good examples of two symmetrical objects are the right and left hand glove which can be made of exactly equal pieces but arranged in reverse order. The right and left hands are also good examples; or an object and its apparent image in a mirror.

64. *Scholium* 2. If each of the face angles BAC, BAD (and also their equals FEG, FEH) is acute, the proof is made very simple by passing planes through B and F perpendicular to the edges AB, EF respectively. Then the angles DBC, HFG are respectively the measures of the diedral angles whose edges are AB, EF. This method cannot be used when the angle BAC, e. g. is not acute, since the perpendicular plane through B would not cut the edge AC.

EXERCISES.

65. If several planes intersect another plane in the same straight line,

1st. The sum of all the diedral angles on one side of this plane is equal to two right diedral angles. (34, 35 ; I. 44.)

2d. The sum of all the diedral angles thus formed is equal to four right diedral angles. (I. 46.)

66. If two planes intersect one another,

1st. Each diedral angle is the supplement of its adjacent diedral angle. (I. 45.)

2d. The vertical diedral angles are equal. (I. 48.)

67. In like manner by changing the word *point* to *straight line*, *straight line* to *plane*, inserting *diedral* before the word *angle*, etc., state and prove, as propositions relating to planes, 47, 49, 54, 55, 56, of Book I.

68. If the face angles that form the polyedral angle A in the Fig. in Art. 60 become equal to four right angles, what then ?

69. Two triedral angles are equal, or symmetrical,

1st. If they have two face angles, and the included diedral angle of the one equal respectively to two face angles and the included diedral angle of the other. (I. 80.)

2d. If they have two diedral angles and the included face angle of the one equal respectively to two diedral angles and the included face angle of the other.

3d. If the three diedral angles of the one are equal respectively to the three diedral angles of the other.

70. Is there anything in plane triangles analogous to (69, 3d) ?

71. State with the necessary changes to make the application to triedrals, 82, 83, 84, 85, 86, 87, 101, 102, of Book I. Prove each proposition.

72. Show that two triedral angles may have three parts of which one is a side, or even four of which two are sides, respectively equal, and be neither equal nor symmetrical.

73. The planes bisecting the diedral angles of a triedral angle intersect in the same straight line; and any point of this line is equally distant from each face of the triedral.

74. The planes passing through the lines that bisect the face angles of a triedral angle, and perpendicular to the faces respectively, intersect in the same straight line; and any point of this line is equally distant from the edges of the triedral.

75. The planes passing through the edges of a triedral angle, and perpendicular to the opposite faces respectively, intersect in the same straight line.

76. The planes passing through the edges and the lines bisecting the face angles of a triedral angle respectively, intersect in the same straight line.

77. If one face of a triedral angle is rectangular, an adjacent diedral angle and its opposite face are both acute, or both right, or both obtuse.

78. If from a point within a triedral angle perpendiculars are drawn to the three faces, these perpendiculars will be the edges of a triedral angle whose face angles will be supplements respectively of the measures of the diedral angles of the given triedral angle. (39, 38; I. 125.)

79. *Corollary.* The face angles of the first triedral angle named in (78) are also supplements of the measures of the diedral angles of the second triedral angle. These triedral angles are called *supplementary triedral angles* of each other.

80. The sum of the diedral angles of a triedral angle is greater than two, and less than six right diedral angles. (60, 79.)

81. If the diedral angles of a triedral angle become equal to two right diedral angles, what then? what, if equal to six?

BOOK VII.

POLYEDRONS.

DEFINITIONS.

1. A **Polyedron** is a solid bounded by planes.

The bounding planes are called *faces;* their intersections, *edges;* the intersections of the edges, *vertices.*

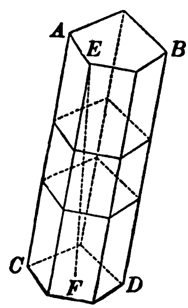
2. A polyedron with four faces is called a *tetraedron;* with six, a *hexaedron;* with eight, an *octaedron;* with twelve, a *dodecaedron;* with twenty, an *icosaedron.*

3. The **Volume** of a solid is the measure of its magnitude. It is expressed in units which represent the number of times it contains the cubical unit taken as a standard.

4. **Equivalent Solids** are those which are equal in volume.

PRISMS AND CYLINDERS.

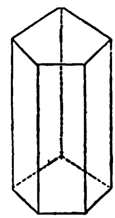
5. A **Prism** is a polyedron two of whose faces are equal polygons having their homologous sides parallel, and whose other faces are parallelograms. *Corollary.* The lateral edges are equal to each other.



The equal parallel polygons are called *bases;* as AB and CD; and the other faces together form the *convex surface.*

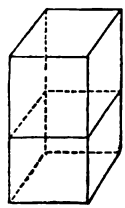
6. The **Altitude** of a prism is the perpendicular distance between its bases; as EF.

7. A **Right Section** of a prism is a section formed by a plane intersecting the prism at right angles to its lateral edges.



8. A **Right Prism** is one whose other faces are perpendicular to its bases. (*Corollary.*) Its lateral faces are rectangles.

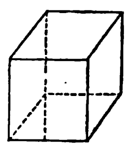
9. A prism is called *triangular*, *quadrangular*, or *pentagonal*, according as its base is a triangle, a quadrangle, or a pentagon; and so on.



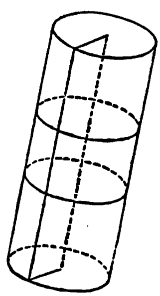
10. A **Parallelopiped** is a prism whose bases are parallelograms. (*Corollary.*) It follows that all its faces are parallelograms, and that any two opposite faces are equal and parallel.

11. A **Right Parallelopiped** is a right prism whose bases are parallelograms. (*Corollary.*) It follows that all its lateral faces are rectangles.

12. A **Rectangular Parallelopiped** is a right parallelopiped whose bases are rectangles.



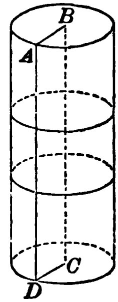
13. A **Cube** is a parallelopiped whose faces are all squares. (*Corollary.*) It follows that its faces are all equal, and the parallelopiped rectangular.



14. A **Cylinder** is a prism whose bases are polygons of an infinite number of sides, that is, whose bases are any closed curves whatever.

15. A **Circular Cylinder** is one whose right section is a circle.

16. A **Right Circular Cylinder** can be described by the revolution of a rectangle about one of its sides which remains fixed. The side opposite the fixed side describes the *convex surface*, and the other two sides the two circular bases. Thus the rectangle $A\,B\,C\,D$ revolving about $B\,C$ would describe the right circular cylinder, the side $A\,D$ the convex surface, and $A\,B$, $D\,C$ the circular bases.

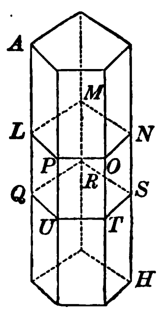


17. The **Axis** of a cylinder is the straight line joining the centres of the two bases. In the right circular cylinder it is the fixed side of the rectangle whose revolution describes the cylinder; as $B\,C$.

THEOREM I.

18. *The sections of a prism made by parallel planes are equal polygons.*

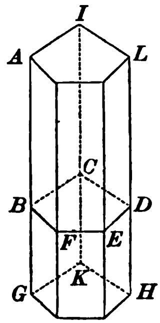
Let the prism $A\,H$ be intersected by the parallel planes $L\,N$ and $Q\,S$; then $L\,N$ and $Q\,S$ are equal polygons. For $L\,M$, $M\,N$, $N\,O$, &c., are respectively parallel to $Q\,R$, $R\,S$, $S\,T$, &c. (VI. 26), and similarly situated; therefore the angles L, M, N, O, P, are respectively equal to the angles Q, R, S, T, U (VI. 28); and the polygons $L\,N$ and $Q\,S$ are mutually equiangular. Also the sides $L\,M$, $M\,N$, $N\,O$, &c., are respectively equal to $Q\,R$, $R\,S$, $S\,T$, &c. (I. 117). Therefore the polygons, being mutually equiangular and equilateral, are equal (II. 34).



19. *Corollary.* A section made by a plane parallel to the base is equal to the base.

THEOREM II.

20. *The convex surface of a prism is equal to the perimeter of a right section of the prism multiplied by a lateral edge.*



Let $B\,C\,D\,E\,F$ be a right section of the prism $A\,H$; its convex surface

$$=(B\,C+C\,D+D\,E+E\,F+F\,B)\times A\,G$$

For the sides of the section $B\,D$ being perpendicular to the edges $A\,G$, $I\,K$, &c., are the altitudes of the parallelograms which form the convex surface of the prism, if we consider the edges $A\,G$, $I\,K$, &c., as bases of these parallelograms. Therefore the convex surface

$$=A\,G\times B\,C+I\,K\times C\,D+L\,H\times D\,E+\text{\&c.}$$

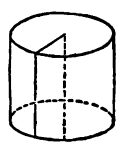
But $$A\,G=I\,K=L\,M=\text{\&c}$$

Therefore the convex surface

$$=(B\,C+C\,D+D\,E+E\,F+F\,B)\times A\,G$$

21. *Cor.* 1. The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.

22. *Cor.* 2. As a cylinder is a prism (14), this demonstration includes the cylinder.

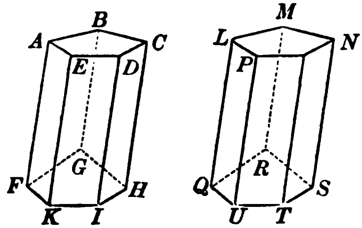


In a circular cylinder if $R=$ the radius of a right section, and $A=$ the axis, the convex surface $=2\,\pi\,R\,A$.

23. *Definition.* A **Truncated Prism** is a part of a prism cut off by a plane not parallel to the base.

THEOREM III.

24. *Two truncated prisms are equal if three faces including a triedral angle of the one are equal respectively to three faces similarly situated, including a triedral angle of the other.*



Let $A\,H$ be a truncated prism whose faces $A\,C$, $A\,G$, $A\,K$, including the triedral angle A, are equal respectively to $L\,N$, $L\,R$, $L\,U$, similarly placed, and including the triedral angle L, of the truncated prism $L\,S$; then the prism $A\,H = L\,S$. For the triedral angles A and L, having their face angles equal and similarly situated, are equal (VI. 62), and can be applied so as to coincide; then the faces $A\,C$, $L\,N$ being equal, and $A\,G$, $L\,R$, and $A\,K$, $L\,U$, will coincide. Now $F\,G$, $F\,K$, coinciding with $Q\,R$, $Q\,U$ respectively, the base $F\,H$ will fall upon the base $Q\,S$ (IV. 2); and as $E\,D$, $E\,K$ coincide respectively with $P\,O$, $P\,U$, the face $E\,I$ must fall upon $P\,T$; and as the bases $F\,H$, $Q\,S$ coincide, $K\,I$ must fall upon $U\,T$, and the face $E\,I$ must coincide with $P\,T$. In like manner it can be proved that the other lateral faces respectively coincide; therefore the prism $A\,H = L\,S$.

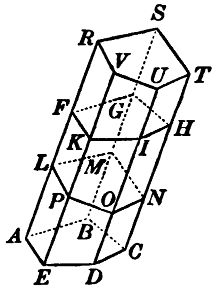
25. *Cor.* 1. The same demonstration applies equally well if the faces $A\,C$, $F\,H$ are parallel; therefore *two prisms are equal if three faces including a triedral angle of the one are equal respectively to three faces similarly situated, including a triedral angle of the other.*

26. *Cor.* 2. If the prisms, $A\,H$, $L\,S$, are right, it is evident that the triedral angle $A = F$, and $L = Q$; and if the faces of the triedral angles A and L are not similarly situated, yet by inverting $A\,H$, the faces of the triedral angles F and L will be similarly situated; and the right prisms can be made to coincide throughout; therefore *two right prisms are equal if three faces including a triedral angle of the one are equal respectively to three faces including a triedral angle of the other.*

27. *Cor.* 3. Two right prisms having equal bases and altitudes are equal.

THEOREM IV.

28. *Any oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.*

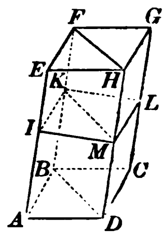


Let $A\,H$ be an oblique prism. Through any point L in the lateral edge $A\,F$, pass a plane perpendicular to $A\,F$, forming the right section $L\,M\,N\,O\,P$. Produce $A\,F$ to R, making $L\,R = A\,F$, and through R pass a plane parallel to $L\,N$. The intersection of this plane with the lateral faces will form a right section equal to $L\,N$ (18). The figure $L\,M\,N\,O\,P-R$ is a right prism whose base is the right section $L\,N$, and whose altitude is $L\,R$, which is equal to $A\,F$, or any lateral edge of the oblique prism $A\,H$.

The figure $A\,B\,C\,D\,E-L$ is a truncated prism which is equal to the truncated prism $F\,G\,H\,I\,K-R$ (24). If we take $A\,B\,C\,D\,E-L$ from the whole figure $A\,B\,C\,D\,E-R$, there remains the right prism $L\,M\,N\,O\,P-R$; and if we take $F\,G\,H\,I\,K-R$ from $A\,B\,C\,D\,E-R$, there remains the oblique prism $A\,B\,C\,D\,E-F$; therefore the oblique prism $A\,B\,C\,D\,E-F$ is equivalent to the right prism $L\,M\,N\,O\,P-R$.

THEOREM V.

29. *The plane passing through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Let the plane $B\,F\,H\,D$ pass through the diagonally opposite edges $B\,F$, $D\,H$, of the parallelopiped $A\,G$; it divides it into two equivalent triangular prisms $A\,B\,D-E$ and $B\,C\,D-G$.

Through any point I in the lateral edge $A\,E$ pass a plane perpendicular to $A\,E$, forming the right section $I\,K\,L\,M$; $I\,K\,L\,M$ is a

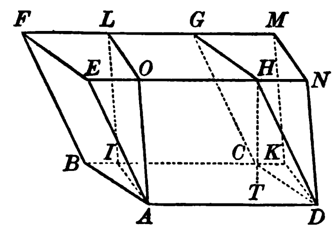
parallelogram (VI. 26), and is divided into two equal triangles by KM, the intersection of the two planes $BFHD$ and $IKLM$.

The oblique prism $ABD-E$ is equivalent to the right prism whose base is IKM and whose altitude is DH (28); and the oblique prism $BCD-G$ is equivalent to the right prism whose base is KLM and whose altitude is DH. But the two right prisms are equal (27); therefore the oblique prisms are equivalent to each other.

30. *Corollary.* A triangular prism is half of a parallelopiped which has the same altitude and a base of twice the area.

THEOREM VI.

31. *Any parallelopiped is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base.*

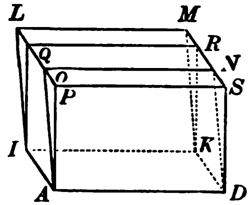


Let AG be a parallelopiped whose bases are $ABCD$ and $EFGH$, and altitude HT.

Through A and D pass planes AL, DM perpendicular to AD; by the intersection of these planes with the faces, and the faces produced, of the given parallelopiped, there will be formed a new parallelopiped, $AIKD-OLMN$, of the same altitude and an equivalent base (II. 44). And the given parallelopiped $ABCD-EFGH$ is equivalent to $AIKD-OLMN$ (28); for taking $ABFE$ as the base of the given parallelopiped, $AILO-DKMN$ is a right prism whose base $AILO$ is a right section of the given oblique prism, and whose altitude is AD, a lateral edge of the given oblique prism.

Now through the edges AD, IK pass planes perpendicular to the base $AIKD$; by the intersection of these planes with

the faces, and the faces produced, of $A\,I\,K\,D - O\,L\,M\,N$, a rectangular parallelopiped $A\,I\,K\,D - P\,Q\,R\,S$ will be formed of the same altitude and base as $A\,I\,K\,D - O\,L\,M\,N$. And by taking $I\,L\,M\,K$ as the base, and $I\,Q\,R\,K$ as a right section, these two parallelopipeds are also by (28) equivalent.



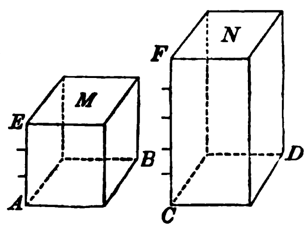
Therefore $A\,B\,C\,D - E\,F\,G\,H$ is equivalent to the rectangular parallelopiped $A\,I\,K\,D - P\,Q\,R\,S$ which has an equivalent base and the same altitude.

THEOREM VII.

32. *Rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Let M and N be rectangular parallelopipeds having equal bases, $A\,B$, $C\,D$, whose altitudes are $A\,E$ and $C\,F$; then

$$M : N = A\,E : C\,F$$



1st. When the altitudes have a common measure which is contained, for example, 3 times in $A\,E$, and 5 times in $C\,F$; then

$$M : N = 3 : 5$$

For, if $A\,E$ is divided into three equal parts, and $C\,F$ into five equal parts, and through the points of division of $A\,E$ and $C\,F$ planes are passed perpendicular to $A\,E$ and $C\,F$, the parallelopiped M will be divided into three, and N into five parallelopipeds, equal each to each (27); therefore

we have $\qquad M : N = 3 : 5$

Hence $\qquad M : N = A\,E : C\,F$

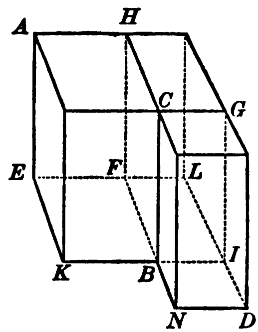
2d. When the altitudes are incommensurable, the proposition is proved by the method adopted in (II. 35).

Therefore, whether the altitudes are commensurable or not,

$$M : N = A\,E : C\,F$$

THEOREM VIII.

33. *Rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let the rectangular parallelopipeds $A\,B$, $C\,D$, have equal altitudes; then

$$A\,B : C\,D = E\,B : B\,D$$

If the parallelopipeds are so placed that the edge $B\,C$ is common, and the right diedral angles at $B\,C$ vertical, and the faces $E\,B$, $A\,F$, $A\,C$, and $D\,G$ are produced, a third rectangular parallelopiped, $H\,I$, will be formed with which $A\,B$ and $C\,D$ can be compared. The rectangular parallelopipeds $A\,B$, $H\,I$, having the same base $C\,F$, are to each other as their altitudes $B\,K$, $B\,I$ (32);

that is $$A\,B : H\,I = B\,K : B\,I$$

In like manner the parallelopipeds $H\,I$, $C\,D$, having the same base $C\,I$, are as their altitudes $B\,F$, $B\,N$;

that is $$H\,I : C\,D = B\,F : B\,N$$

Multiplying these two proportions together, we have (II. 21, 24)

$$A\,B : C\,D = B\,K \times B\,F : B\,I \times B\,N$$

But $B\,K \times B\,F$ is the area of the base $E\,B$, and $B\,I \times B\,N$ of the base $B\,D$; therefore

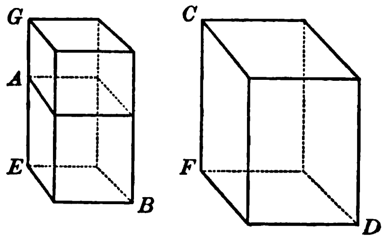
$$A\,B : C\,D = E\,B : B\,D$$

THEOREM IX.

34. *Rectangular parallelopipeds are to each other as the products of their bases by their altitudes.*

Let $A B$, $C D$, be rectangular parallelopipeds, then

$$A B : C D = E B \times E A : F D \times F C$$



Produce the edge $E A$ to G making $E G$ equal to $F C$; if through G a plane is passed parallel to $E B$ meeting the edges of $A B$, produced, a rectangular parallelopiped $G B$ will be formed with the same altitude as $C D$ and the same base as $A B$.

Hence (32) $\qquad A B : G B = E A : E G$

and (33) $\qquad G B : C D = E B : F D$

Multiplying these two proportions together, and substituting $F C$ for its equal $E G$, we have

$$A B : C D = E B \times E A : F D \times F C$$

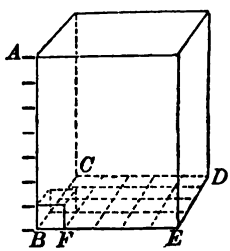
35. *Cor.* 1. Hence, any rectangular parallelopiped is to the cubical unit taken as a standard, as the product of its base by its altitude is to unity; therefore *the volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

36. *Cor.* 2. As the area of a rectangle is equal to the product of its two dimensions, *the volume of a rectangular parallelopiped is equal to the product of its three dimensions.*

37. *Cor.* 3. The volume of a cube is equal to the cube of one of its edges.

38. *Scho.* 1. By a *product of the base by the altitude* is meant the product of the numbers which express the number of square units in the base and the number of linear units in the altitude respectively. By *product of the three dimensions* is meant the product of the numbers which express the number of linear units in the three dimensions respectively. By *cube of an edge* is meant the third power of the number which expresses the number of linear units in the edge.

39. *Scho.* 2. When the three dimensions have a common measure this proposition is made evident by dividing the three dimensions into equal parts representing the linear units. Suppose BF, the linear unit, is contained in BC four times, in BE five times, and in BA seven times; then dividing BC, BE, BA respectively into four, five, and seven equal parts, and passing planes through the several points of division parallel to the sides of the parallelopiped, there will be formed a number of cubes equal to each other (27), and each equal to the cube whose edge is the linear unit. It is evident also that the whole number of cubes is equal to the product of the three dimensions, or $4 \times 5 \times 7 = 140$; that is, the volume of $AD = 4 \times 5 \times 7 = 140$.



THEOREM X.

40. *The volume of any prism is equal to the product of its base by its altitude.*

1st. Any parallelopiped is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base (31); and the volume of the equivalent rectangular parallelopiped is equal to the product of its base by its altitude (35); therefore the volume of any parallelopiped is equal to the product of its base by its altitude.

2d. A triangular prism is half of a parallelopiped which has the same altitude and a base of twice the area (30); and the volume of this parallelopiped is equal to the product of its base by its altitude; therefore the volume of a triangulur prism is equal to the product of its base by its altitude.

3d. By passing planes through its lateral edges, any prism can be divided into triangular prisms, whose altitudes are the same as the altitude of the prism, and whose triangular bases together form the base of the prism; and the volume of each of these triangular prisms is equal to the product of its base by its altitude; therefore the volume of any prism is equal to the product of its base by its altitude.

41. *Cor.* 1. Prisms having equivalent bases are as their altitudes; prisms having equal altitudes are as their bases; and prisms are as the products of their bases by their altitudes.

42. *Cor.* 2. As a cylinder is a prism, this demonstration includes the cylinder. If, therefore, $R =$ the radius of base, $A =$ the altitude, and $V =$ the volume of a cylinder,

$$V = \pi R^2 A = \tfrac{1}{4} \pi D^2 A$$

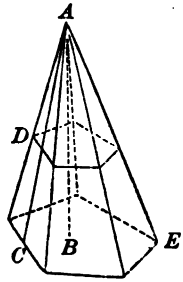
PYRAMIDS AND CONES.

DEFINITIONS.

43. A **Pyramid** is a polyedron bounded by a polygon called the base, and by triangular planes meeting at a common point called the vertex.

44. A pyramid is called *triangular*, *quadrangular*, *pentagonal*, according as its base is a triangle, a quadrangle, or a pentagon; and so on.

45. A **Tetraedron** has but four faces all triangles, and is a *triangular pyramid.* Any of its faces may be taken as a base.



46. The **Altitude** of a pyramid is the perpendicular distance from its vertex to its base; as $A\ B$.

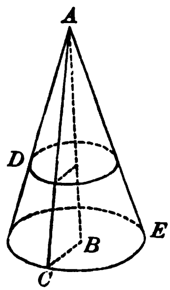
47. A **Right Pyramid** is one whose base is a regular polygon, and in which the perpendicular from the vertex passes through the centre of the base.

48. The **Slant Height** of a right pyramid is the perpendicular distance from the vertex to the base of any one of its lateral faces; as $A\ C$.

49. A **Cone** is a pyramid whose base is a polygon of an infinite number of sides, that is, whose base is any closed curve whatever.

50. A **Circular Cone** is one whose right section, that is, the section formed by a plane perpendicular to the axis, is a circle.

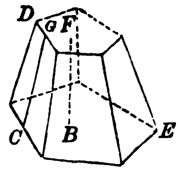
51. A **Right Circular Cone** can be described by the revolution of a right triangle about one of its sides which remains fixed. The other side describes the circular base, and the hypothenuse the *convex surface.* Thus the right triangle $A\ B\ C$ revolving about $A\ B$ would describe the right circular cone, $B\ C$ the base, and the hypothenuse $A\ C$ the convex surface.



52. The **Axis** of a cone is the line from the vertex to the centre of the base. In the right circular cone it is the fixed side of the right triangle whose revolution describes the cone; as $A\ B$.

53. *Corollary.* The axis of a right cone is perpendicular to the base, and is therefore the *altitude* of the cone.

54. A **Frustum** of a pyramid is a part of the pyramid included between the base and a plane cutting the pyramid parallel to the base; as DE.



55. The **Altitude** of a frustum is the perpendicular distance between the two parallel planes or bases; as FB.

56. The **Slant Height** of a frustum of a right pyramid is the perpendicular distance between the parallel edges of the bases; as GC.

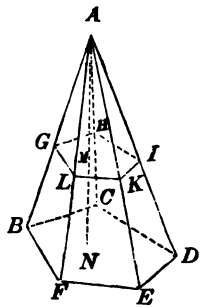
THEOREM XI.

57. *If a pyramid is cut by a plane parallel to its base,*

1st. *The edges and altitude are divided proportionally;*

2d. *The section is a polygon similar to the base.*

Let $A-BCDEF$ be a pyramid whose altitude is AN, cut by a plane GI parallel to the base; then



1st. The edges and the altitude are divided proportionally.

For suppose a plane passed through the vertex A parallel to the base; then the edges and altitude, being cut by three parallel planes, are divided proportionally (VI. 29), and we have

$$AB : AG = AC : AH = AD : AI = AN : AM$$

2d. The section GI is similar to the base BD.

For the sides of GI are respectively parallel to the sides of BD (VI. 26), and similarly situated; therefore the polygons GI, BD are mutually equiangular. Also, as GL is parallel to BF,

and LK to FE, the triangles ABF and AGL are similar, and the triangles AFE and ALK; therefore

$$GL : BF = AL : AF, \text{ and } LK : FE = AL : AF$$

Therefore $$GL : BF = LK : FE$$

In the same manner we should find

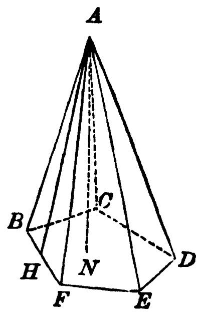
$$LK : FE = KI : ED = IH : DC, \text{ \&c.}$$

Therefore the polygons GI and BD are similar (II. 76).

58. *Corollary.* A section of a right circular cone made by a plane parallel to the base is a circle.

THEOREM XII.

59. *The convex surface of a right pyramid is equal to the perimeter of its base multiplied by half its slant height.*



Let A–$BCDEF$ be a right pyramid whose slant height is AH; its convex surface is equal to $BC + CD + DE + EF + FB$ multiplied by half of AH.

The edges AB, AC, AD, AE, AF, being equally distant from the perpendicular AN (II. 82), are equal (IV. 7); and the bases BC, CD, DE, &c. are equal; therefore the isosceles triangles ABC, ACD, ADE, &c. are all equal (I. 88); and their altitudes are equal. The area of ABC is $BC \times \frac{1}{2} AH$ (II. 11); of ACD is $CD \times \frac{1}{2} AH$; and so on. Therefore the sum of the areas of these triangles, that is, the convex surface of the right pyramid, is $(BC + CD + DE + EF + FB) \frac{1}{2} AH$.

60. *Corollary.* As a cone is a pyramid (48), this demonstration includes the right cone. If, therefore, $R =$ the radius of the base, and $S =$ the slant height of a right circular cone,

$$\text{its convex surface} = 2 \pi R \tfrac{1}{2} S = \pi R S$$

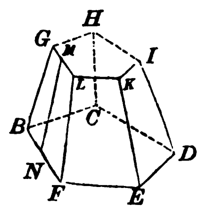
If a plane parallel to the base and bisecting the altitude be

drawn, as the section will be a circle (58) with a radius and circumference one half the radius and circumference of the base, therefore, if $r' =$ the radius of this section,

$$\text{the convex surface} = 2\,\pi\,r'\,S$$

THEOREM XIII.

61. *The convex surface of a frustum of a right pyramid is equal to the sum of the perimeter of its two bases multiplied by half its slant height.*



Let $G\,D$ be the frustum of a right pyramid; its convex surface is equal to $G\,H + H\,I + I\,K + K\,L + L\,G + B\,C + C\,D + D\,E + E\,F + F\,B$ multiplied by half $M\,N$.

The lateral faces of a frustum of a right pyramid are equal trapezoids (57; II. 34); and their altitudes are all equal. The area of $G\,C$ (II. 48) is $(GH + BC) \times \frac{1}{2}MN$; of $H\,D$ is $(H\,I + C\,D) \times \frac{1}{2}M\,N$; and so on. Therefore the sum of the areas of these trapezoids, that is, the convex surface of the frustum of the right pyramid, is $G\,H + H\,I + I\,K + K\,L + L\,G + B\,C + C\,D + D\,E + E\,F + F\,B$ multiplied by half $M\,N$.

62. *Cor.* 1. If the frustum is cut by a plane parallel to its two bases, and at equal distances from each base, this plane will bisect the edges $G\,B$, $H\,C$, $I\,D$, &c. (57); and the area of each trapezoid is equal to its altitude multiplied by the line joining the middle points of the sides which are not parallel (II. 49). Therefore the convex surface of a frustum of a right pyramid is equal to the perimeter of a section midway between the bases multiplied by its slant height.

63. *Cor.* 2. As a cone is a pyramid (49), this demonstration includes the frustum of a right cone. If, therefore, R and

$r =$ the radii of the two bases of the frustum of a right circular cone, and $S =$ its slant height,

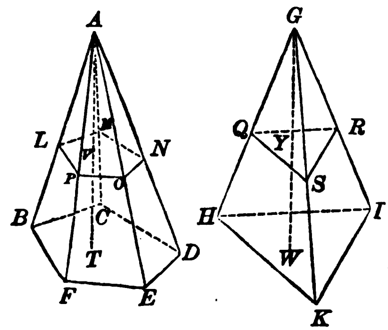
its convex surface $= (2\pi R + 2\pi r)\frac{1}{2}S = (\pi R + \pi r)S$

If $r' =$ the radius of a section midway between and parallel to the bases,

$$\text{the convex surface} = 2\pi r' S$$

THEOREM XIV.

64. *If two pyramids having equal altitudes are cut by planes parallel to their bases and at equal distances from their vertices, the sections are to each other as their bases.*



Let $A - BCDEF$ and $G - HIK$ be two pyramids of equal altitudes AT, GW, cut by the planes $LMNOP$ and QRS parallel respectively to the bases and at equal distances from the vertices A and G, then

$$LMNOP : QRS = BCDEF : HIK$$

For as the polygons $LMNOP$ and $BCDEF$ are similar (57)

$$LMNOP : BCDEF = \overline{LP}^2 : \overline{BF}^2 = \overline{AL}^2 : \overline{AB}^2 = \overline{AV}^2 : \overline{AT}^2$$

In like manner

$$QRS : HIK = \overline{GY}^2 : \overline{GW}^2$$

But as $AV = GY$ and $AT = GW$

therefore

$$LMNOP : BCDEF = QRS : HIK$$

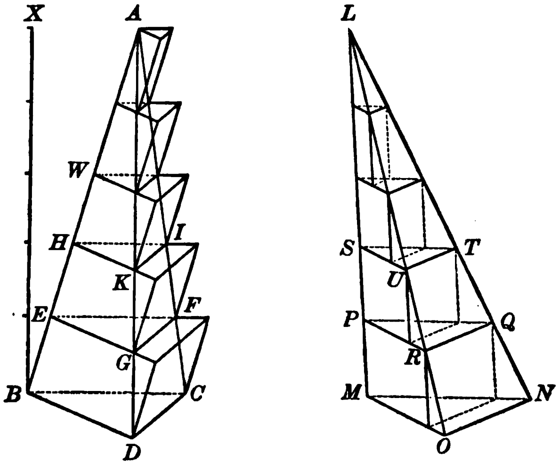
or (II. 15)

$$LMNOP : QRS = BCDEF : HIK$$

65. *Corollary.* If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.

THEOREM XV.

66. *Triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Let $A-BCD$ and $L-MNO$ be triangular pyramids having equivalent bases BCD, MNO, and altitudes equal to BX; then

$$A-BCD = L-MNO$$

Place the pyramids so that BX shall be the common altitude; then their bases will be in the same plane (VI. 11). Divide the common altitude, BX, into any number of equal parts, and through the points of division pass planes parallel to the plane of the bases. The corresponding sections thus formed of the pyramids are equivalent (65); that is, the triangles $EFG = PQR$, $HIK = STV$, &c.

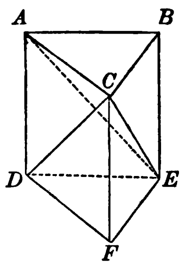
On the triangles BCD, EFG, &c., as lower bases, construct prisms whose lateral edges are parallel to BA, and whose altitudes are each equal to the parts of BX; and on the triangles PQR, STV, &c., as upper bases, construct prisms whose lateral edges are parallel to ML, and whose altitudes are each equal to the parts of BX.

Now the prism $H-EFG$ is equivalent to $PQR-M$ (40); and the prism $W-HIK$ to $STV-P$, and so on; that is, each of the triangular prisms constructed on the sections of $A-BCD$ as lower bases (the prism $E-BCD$ is not constructed on one of these sections) has an equivalent prism constructed on the corresponding sections of $L-MNO$ as an upper base; therefore the sum of all the prisms circumscribed about $A-BCD$ differs from the sum of the prisms inscribed in $L-MNO$ by the prism $E-BCD$. But as the sum of all the prisms circumscribed about $A-BCD$ is greater than the pyramid $A-BCD$, and the sum of all the prisms inscribed in $L-MNO$ is less than the pyramid $L-MNO$, the pyramids differ in volume by a quantity less than the prism $E-BCD$; and this is true without regard to the number of parts into which the altitude BX is divided. Now if the altitude BX is divided into an infinite number of parts, the prism $E-BCD$ will be infinitesimal; but the difference in volume of the pyramids is still less, that is, is zero; or, in other words, the triangular pyramid

$$A-BCD = L-MNO$$

THEOREM XVI.

67. *A triangular pyramid is one third of a triangular prism of the same base and altitude.*



Let $C-DEF$ be a triangular pyramid and $ABC-DEF$ be a triangular prism on the same base DEF; then $C-DEF$ is one third of $ABC-DEF$.

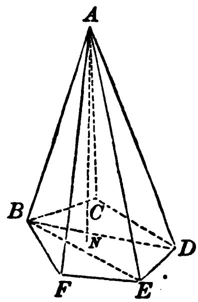
Taking away the pyramid $C-DEF$ there remains the quadrangular pyramid whose vertex is C, and base the parallelogram $ABED$. Through the points A, C, E pass a plane; it will divide the pyramid $C-ABED$ into two triangular pyramids, which are equivalent to each other (66), since their bases are halves of the parallelogram $ABED$, and they have the

same altitude, the perpendicular from their vertex C to the base $A\,B\,E\,D$. But the pyramid $C-A\,B\,E$, that is, $E-A\,B\,C$, is equivalent to the pyramid $C-D\,E\,F$, as they have equal bases $A\,B\,C$ and $D\,E\,F$, and the same altitude (66). Therefore the three pyramids are equivalent and the given pyramid is one third of the prism.

68. *Corollary.* The volume of a triangular pyramid is equal to one third the product of its base by its altitude.

THEOREM XVII.

69. *The volume of any pyramid is equal to one third of the product of its base by its altitude.*



Let $A-B\,C\,D\,E\,F$ be any pyramid; its volume is equal to one third the product of its base $B\,C\,D\,E\,F$ by its altitude $A\,N$.

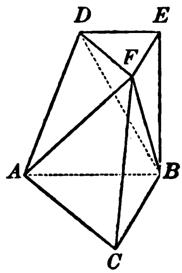
Planes passing through the vertex A and the diagonals of the base $B\,D$, $B\,E$, will divide the pyramid into triangular pyramids whose bases together compose the base of the given pyramid and which have as their common altitude $A\,N$, the altitude of the given pyramid. The volume of the given pyramid is equal to the sum of the volumes of the triangular pyramids, which is equal to one third of the sum of their bases multiplied by their common altitude; that is, is equal to one third of the product of the base $B\,C\,D\,F\,E$ by the altitude $A\,N$.

70. *Cor.* 1. Pyramids having equivalent bases are as their altitudes; pyramids having equal altitudes are as their bases; and pyramids are as the products of their bases by their altitudes.

71. *Cor.* 2. As a cone is a pyramid (49), this demonstration includes the cone. A cone, therefore, is one third of a cylinder, or of any prism, of equivalent base and the same altitude. If $R =$ radius of the base, $A =$ the altitude, and $V =$ the volume of a right circular cone, $V = \frac{1}{3}\,\pi\,R^2\,A$.

THEOREM XVIII.

72. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.*



Let $ABC-DEF$ be a frustum of a triangular pyramid.

Through A, F, B pass a plane; it will cut off a pyramid $F-ABC$ which we will call P, whose altitude is the altitude of the frustum and whose base is ABC, the lower base of the frustum.

Through D, B, F pass a plane; it will cut off a pyramid, $B-DEF$, which we will call Q, whose altitude is the altitude of the frustum, and whose base is DEF, the upper base of the frustum.

There remains the triangular pyramid, which we will call R, whose vertex is F and base ADB.

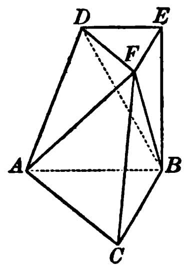
The pyramids P and R regarded as having the common vertex at B, have the same altitude, namely, the perpendicular distance from B to the plane $ADFC$, and therefore are to one another as their bases AFC, ADF (70); but the triangles AFC, ADF, having the same altitude, are as their bases AC, DF (II. 47); that is,

$$P:R = AFC:ADF = AC:DF$$

The pyramids R and Q regarded as having the common vertex F have the same altitude, namely, the perpendicular distance from F to the plane $ADEB$, and therefore are to one another as their bases ADB, DEB; but the triangles ADB, DEB having the same altitude are as their bases AB to DE; that is,

$$R:Q = ADB:DEB = AB:DE$$

But the triangles ABC, DEF are similar (57);



hence $AC : DF = AB : DE$

Therefore $P : R = R : Q$

or $R^2 = P \times Q$, and $R = \sqrt{P \times Q}$

Now let a denote the altitude of the frustum, B the lower base, and b the upper base; then (68)

$$P = \tfrac{1}{3}\, a \times B, \text{ and } Q = \tfrac{1}{3}\, a \times b$$

and $$R = \sqrt{\tfrac{1}{3}\, a \times B \times \tfrac{1}{3}\, a \times b} = \tfrac{1}{3}\, a \sqrt{B \times b}$$

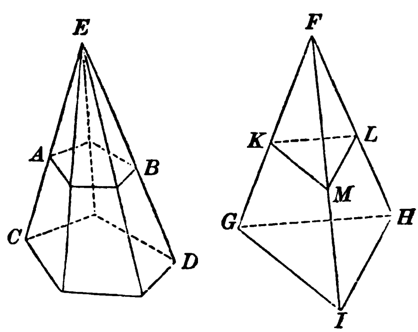
that is, R is equivalent to a pyramid whose altitude is a, the altitude of the prism, and whose base is a mean proportional between B and b, the two bases of the frustum.

73. *Corollary.* If V equal the volume of the frustum of the triangular pyramid, we have

$$V = \tfrac{1}{3}\, a \times B + \tfrac{1}{3}\, a \times b + \tfrac{1}{3}\, a \sqrt{B \times b}$$
$$= \tfrac{1}{3}\, a\,(B + b + \sqrt{B \times b})$$

THEOREM XIX.

74. *A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.*



Let $AB-CD$ be a frustum of a pyramid whose vertex is E.

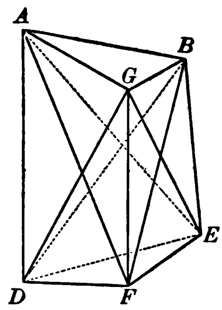
Let $F-GHI$ be a triangular pyramid having the base GHI equivalent to CD and the same altitude as $E-CD$; the pyramid $E-CD$ is equivalent to

$F-GHI$ (69). At the same distance from the vertex F as AB is from E pass a plane parallel to GHI forming the section KLM. $KLM = AB$ (65), and the small pyramid $F-KLM = E-AB$ (69); therefore, taking away these small pyramids from each pyramid, the frustum $AB-CD$ is equivalent to the frustum $KLM-GHI$, that is, is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.

75. *Corollary.* As a cone is a pyramid (49), this demonstration includes the frustum of a cone.

THEOREM XX.

76. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section.*



Let ABC be the inclined section of the truncated triangular prism $ABC-DEF$. Pass the planes ACE, DCE dividing the truncated prism into the three pyramids $C-DEF$, $C-AED$, and $C-ABE$.

The first of these pyramids $C-DEF$ has the base DEF and the vertex C.

Pass the plane FAE so as to form the pyramid $F-AED$. Now the second pyramid $C-AED$ is equivalent to $F-AED$, as they have the same base AED, and the same altitude, since their vertices are in CF which is parallel to the base AED. But $F-AED$ is the same as $A-DEF$; hence the second pyramid $C-AED$ is equivalent to $A-DEF$.

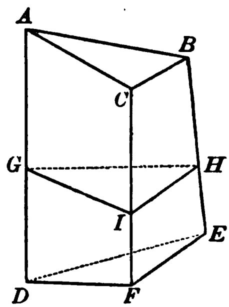
Again, pass the plane BFD so as to form the pyramid

$F-BED$. The third pyramid $C-ABE$ is equivalent to $F-BED$; for, as their vertices are in CF which is parallel to the plane of their bases, they have the same altitude; and they have equivalent bases, since the triangles ABE, DBE, having the same base BE, and the same altitude, are equivalent. But $F-BED$ is the same as $B-DEF$; hence the third pyramid $C-ABE$ is equivalent to $B-DEF$.

Therefore the truncated triangular prism $ABC-DEF$ is equivalent to three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section.

77. *Cor.* 1. If the truncated triangular prism is *right*, the three lateral edges will be the altitudes of the three pyramids to which the truncated triangular prism is equivalent; therefore *the volume of a truncated right triangular prism is equal to the product of its base by one third of the sum of its lateral edges.*

78. *Cor.* 2. *The volume of any truncated triangular prism is equal to the product of a right section by one third of the sum of its lateral edges.*



Let GHI be a right section of the truncated prism $ABC-DEF$.

By (77)

$$ABC-GHI = GHI \times \tfrac{1}{3}(AG + BH + CI)$$

and

$$GHI-DEF = GHI \times \tfrac{1}{3}(GD + HE + IF)$$

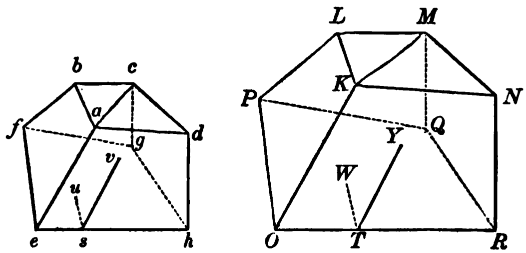
Adding these two equations we have

$$ABC-DEF = GHI \times \tfrac{1}{3}(AD + \quad E + CF)$$

79. *Definition.* **Similar Solids** are those whose homologous lines, that is, lines similarly situated, have a constant ratio. *Corollary.* It follows that similar solids are bounded by the same number of similar polygons similarly situated.

THEOREM XXI.

80. *In similar polyedrons the homologous diedral angles are equal.*

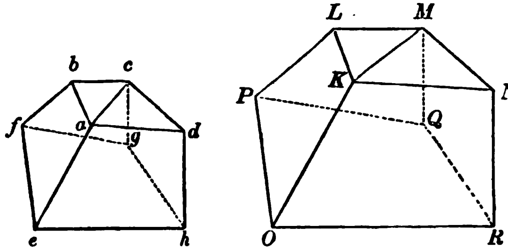


Let $a\,g$ and $K\,Q$ be similar polyedrons; their homologous diedral angles are equal.

Through the homologous points s T of the homologous edges $e\,h$, $O\,R$ pass planes perpendicular to the edges $e\,h$ and $O\,R$ respectively, intersecting the faces $e\,g$, $e\,d$, $O\,Q$, $O\,N$, in the lines $s\,u$, $s\,v$, $T\,W$, $T\,Y$, respectively. The plane angles $u\,s\,v$, $W\,T\,Y$ are respectively the measures of the diedral angles at the edges $e\,h$, $O\,R$ (VI. 34). Let u, v, W, Y be homologous points in their respective planes; join $u\,v$, $W\,Y$. As the homologous sides of the triangles $u\,s\,v$, $W\,T\,Y$ have a constant ratio (79), the triangles are similar (II. 59); hence the homologous angles $u\,s\,v$, $W\,T\,Y$ are equal; therefore the diedral angles formed at the homologous edges $e\,h$ and $O\,R$ are equal. In like manner the other diedral angles of $a\,g$ can be proved equal to the homologous diedral angles of $K\,Q$.

THEOREM XXII.

81. *In similar polyedrons the homologous polyedral angles are equal.*



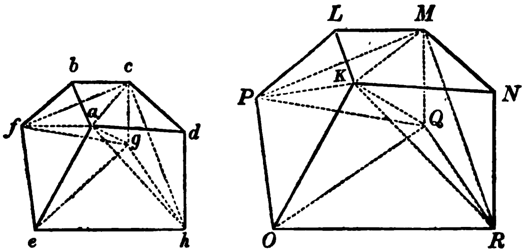
Let $a\,g$ and $K\,Q$ be similar polyedrons; their homologous polyedral angles are equal.

Place the face $a\,b\,c$ on the homologous face $K\,L\,M$ with the vertex of the plane angle at a on K, and the line $a\,b$ on $K\,L$; as the plane angles $b\,a\,c$, $L\,K\,M$ are equal, $a\,c$ will fall on $K\,M$. Now as the diedral angles whose edges are $a\,c$ and $K\,M$ are equal (80), the face $a\,c\,d$ will fall on $K\,M\,N$; and as the plane angles $c\,a\,d$ and $M\,K\,N$ are equal, the edge $a\,d$ will fall on $K\,N$; in like manner the face $a\,h$ will fall on $K\,R$, and the edge $a\,e$ on $K\,O$; and the faces of the polyedral angle a will coincide with the homologous faces of the polyedral angle K; therefore the polyedral angles a and K are equal. In like manner the other polyedral angles of $a\,g$ can be proved equal to the homologous polyedral angles of $K\,Q$.

THEOREM XXIII.

82. *Similar polyedrons can be divided into the same number of similar tetraedrons.*

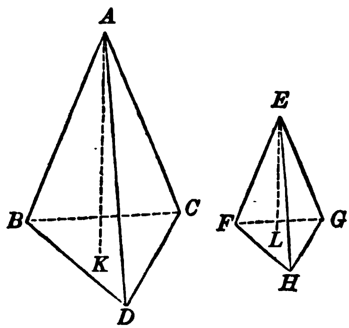
Let $a\,g$ and $K\,Q$ be similar polyedrons: they can be divided into the same number of similar tetraedrons. From the homologous vertices a and K draw diagonals to all the vertices of $a\,g$ and $K\,Q$ respectively, and in each of the faces of $a\,g$ and $K\,Q$



not adjacent to a and K draw homologous diagonals dividing the homologous faces into similar triangles; the polyedrons will be divided into similar tetraedrons whose bases are these triangles and whose vertices are at a and K. For, as the polyedrons are similar, the homologous lines in the homologous tetraedrons must have the same constant ratio.

THEOREM XXIV.

83. *Similar polyedrons are as the cubes of their homologous lines.*



1st. Let BCD and FGH be the homologous faces of the similar tetraedrons $A-BCD$ and $E-FGH$, and AK, EL the homologous altitudes.

Let V represent the volume of $A-BCD$ and v the volume of $E-FGH$; then

$$V : v = \overline{BD}^3 : \overline{FH}^3$$

For (70) $\quad V : v = BCD \times AK : FGH \times EL$

But (II. 79) $\quad BCD : FGH = \overline{BD}^2 : \overline{FH}^2$

and (79) $\quad AK : EL = BD : FH$

Multiplying the last two proportions together, we have

$$BCD \times AK : FGH \times EL = \overline{BD}^3 : \overline{FH}^3$$

therefore $$V : v = \overline{BD}^3 : \overline{FH}^3$$

But in similar solids homologous lines have a constant ratio (79); therefore $V : v$ as the cubes of any homologous lines.

2d. Similar polyedrons can be divided into the same number of similar tetraedrons (82); and any two of these similar tetraedrons are as the cubes of their homologous lines which are also homologous lines of the polyedrons; but the ratio of the homologous lines of the polyedrons is constant; therefore the sums of the tetraedrons, that is, the polyedrons themselves, are as the cubes of their homologous lines.

REGULAR POLYEDRONS.

84. *Definition.* A **Regular Polyedron** is a polyedron whose faces are all equal regular polygons and whose polyedral angles are equal each to each.

THEOREM XXV.

85. *There can be but five regular polyedrons.*

The sum of the face angles of a polyedral angle must be less than four right angles (VI. 60); therefore

1st. Each polyedral angle may be included by three, or four, or five, but not more, equilateral triangles; for each angle of an equilateral triangle is equal to two thirds of a right angle (I. 84, 78), and

$3 \times \frac{2}{3}$ right angle $<$ 4 right angles
$4 \times \frac{2}{3}$ " " $<$ 4 " "
$5 \times \frac{2}{3}$ " " $<$ 4 " "
$6 \times \frac{2}{3}$ " " $=$ 4 " "

2d. By three and no more squares; for each angle of a square is a right angle, and

3×1 right angle $<$ 4 right angles
4×1 " " $=4$ " "

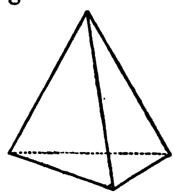
3d. By three, and no more, regular pentagons; for each angle of a regular pentagon is equal to six fifths of a right angle (I. 125), and

$3 \times \frac{6}{5}$ right angle $<$ 4 right angles
$4 \times \frac{6}{5}$ " " >4 " "

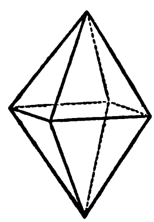
A polyedral angle cannot be included by regular hexagons; for as each angle of a regular hexagon is equal to four thirds of a right angle (I. 125), three such face angles would contain four right angles. Therefore a polyedron cannot have as its faces regular polygons of more than five sides.

Therefore we can only have five regular polyedrons; three whose faces are equilateral triangles, one whose faces are squares, and one whose faces are regular pentagons.

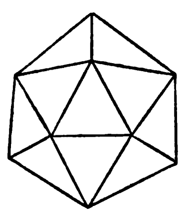
86. *Sch.* 1. When each polyedral angle is included by three equilateral triangles, the figure is a *regular tetraedron*, or a right triangular pyramid whose faces and base are equal.



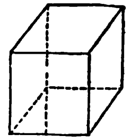
When each polyedral angle is included by four equilateral triangles, the figure is a *regular octaedron.*



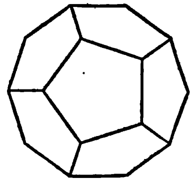
When each polyedral angle is included by five equilateral triangles, the figure is a *regular icosaedron.*



When each polyedral angle is included by squares, the figure is a *regular hexaedron*, or *cube*.

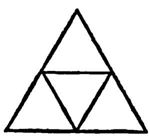


When each polyedral angle is included by pentagons, the figure is a *regular dodecaedron.*

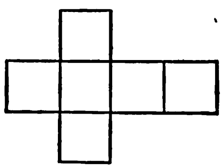


87. *Sch.* 2. Models of these five regular polyedrons can be constructed by drawing on card-board the following diagrams; then cut them out entire, and at the lines separating the polygons cut the card-board half through; the edges can now be brought together and the models will be formed.

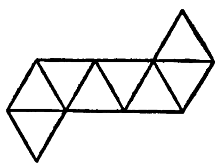
REG. TETRAEDRON.



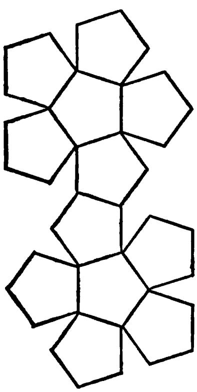
REG. HEXAEDRON.



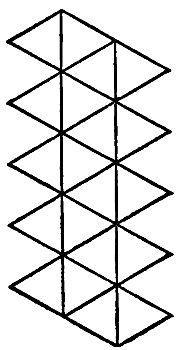
REG. OCTAEDRON.



REG. DODECAEDRON.



REG. ICOSAEDRON.



THE SPHERE.

DEFINITIONS.

88. A **Sphere** is a solid bounded by a curved surface, of which every point is equally distant from a point within called the *centre*. A sphere can be described by the revolution of a semicircle about its diameter which remains fixed.

89. The **Radius** of a sphere is the straight line from the centre to any point of the surface.

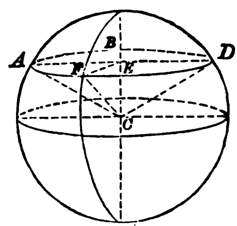
90. The **Diameter** of a sphere is a straight line passing through the centre and terminating at either end at the surface.

91. *Corollary.* All the radii of a sphere are equal; all the diameters are equal, and each is double the radius.

THEOREM XXVI.

92. *Every section of a sphere made by a plane is a circle.*

Let ABD be a section made by a plane cutting the sphere whose centre is C; then is ABD a circle.



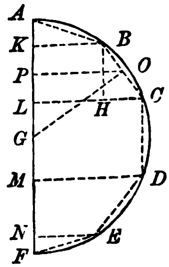
Draw CE perpendicular to the plane, and to the points A, D, F, where the plane cuts the surface of the sphere, draw CA, CD, CF. As CA, CD, CF are radii of the sphere they are equal, and are therefore equally distant from the foot of the perpendicular CE (VI. 8). Therefore EA, ED, EF are equal, and the section ABD is a circle whose centre is E.

93. *Corollary.* If the section passes through the centre of the sphere, its radius will be the radius of the sphere.

94. *Definition.* A section made by a plane passing through the centre of a sphere is called a *great circle*. A section made by a plane not passing through the centre is called a *small circle*.

THEOREM XXVII.

95. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



Let $ABCDEF$ be the semicircle by whose revolution about the diameter AF, the sphere may be described; then the surface of the sphere is equal to the diameter AF multiplied by the circumference of the circle whose radius is GA, or $= AF \times$ circ. GA.

Let $ABCDEF$ be a regular semi-decagon inscribed in the semicircle. Draw GO perpendicular to one of its sides, as BC.

Draw BK, OP, CL, DM, EN perpendicular to the diameter AF, and BH perpendicular to CL. The surface described by BC is the convex surface of the frustum of a cone, and is equal to $BC \times$ circ. PO (63). But the triangles BCH and POG are similar (II. 56); therefore

$$BC : BH \text{ or } KL = GO : PO$$

or (III. 45) $$BC : KL = \text{circ. } GO : \text{circ. } PO$$

$$\therefore BC \times \text{circ. } PO = KL \times \text{circ. } GO$$

That is, the surface described by BC is equal to the altitude KL multiplied by circ. GO, or by the circumference of the circle inscribed in the polygon. In like manner it can be proved that the surfaces described by AB, CD, DE, and EF are respectively equal to their altitudes AK, LM, MN, and NF multiplied by circ. GO. Therefore the entire surface described by the semi-polygon will be equal to

$$(AK + KL + LM + MN + NF) \text{ circ. } GO = AF \times \text{circ. } GO$$

This demonstration is true, whatever the number of sides of [illegible]he semi-polygon; it is true, therefore, if the number of sides [illegible] which case the semi-polygon would coincide with [illegible] and the surface described by the semi-polygon [illegible]urface of the sphere, and the radius of the in-

scribed polygon would be the radius of the sphere. Therefore we have the surface of the sphere equal to

$$A\,F \times \text{circ. } G\,A$$

96. *Corollary.* Let $S =$ the surface of the sphere, $C =$ the circumference, $R =$ the radius, $D =$ the diameter, then we have (III. 30) $\quad C = 2\,\pi\,R$, or $\pi\,D$

Therefore $\quad S = 2\,\pi\,R \times 2\,R = 4\,\pi\,R^2$, or $\pi\,D^2$

That is, *the surface of a sphere is equal to the square of its diameter multiplied by* 3.14159.

THEOREM XXVIII.

97. *The volume of a sphere is the product of its surface by one third of its radius.*

A sphere may be conceived to be composed of an infinite number of pyramids whose vertices are at the centre of the sphere, and whose bases, being infinitely small planes, coincide with the surface of the sphere. The altitude of each of these pyramids is the radius of the sphere, and the sum of the surfaces of their bases is the surface of the sphere. The volume of each pyramid is the product of the area of its base by one third of its altitude, that is, of the radius of the sphere (69); and the volume of all the pyramids, that is, of the sphere, is, therefore, the product of the surface of the sphere by one third of its radius.

98. *Cor.* 1. Let $V =$ the volume of the sphere, and R, D, and S the same as in (96). Then, as (96)

$$S = 4\,\pi\,R^2, \text{ or } \pi\,D^2$$

$$V = 4\,\pi\,R^2 \times \tfrac{1}{3}\,R = \tfrac{4}{3}\,\pi\,R^3, \text{ or } \tfrac{1}{6}\,\pi\,D^3$$

That is, *the volume of a sphere is the cube of the diameter multiplied by* .5236.

99. *Cor.* 2. As in these equations $\frac{4}{3}\,\pi$ and $\frac{1}{6}\,\pi$ are constant, *the volumes of spheres vary as the cubes of their radii, or as the cubes of their diameters.*

PRACTICAL QUESTIONS.

1. How many square feet in the convex surface of a right prism whose altitude is 2 feet, and whose base is a regular hexagon of which each side is 8 inches long? How many square feet in the whole surface?

2. The radius of the base of a cylinder is 6 inches, and its altitude 3 feet; how many square feet in the whole surface?

3. What is the number of feet in the bounding planes of a cube whose edge is 5 feet? The number of solid feet in the cube?

4. What is the number of feet in the bounding planes of a right parallelopiped whose three dimensions are 4, 7, and 9 feet? The number of cubic feet in the parallelopiped?

5. What is the number of cubic feet in the right prism whose dimensions are given in the first example?

6. What is the number of cubic feet in the cylinder whose dimensions are given in the second example?

7. The altitude of a prism is 9 feet and the perimeter of the base 6 feet. What is the altitude and perimeter of the base of a similar prism one third as great?

8. What is the ratio of the volumes of two cylinders whose altitudes are as 3 : 6, if the cylinders are similar? What, if the bases are equal? What, if the bases are as 3 : 6 and the altitudes equal?

9. How many square feet in the convex surface of a right pyramid whose slant height is 3 feet, and whose base is a regular octagon of which each side is 2 feet long?

10. How many square feet in the convex surface of a cone whose slant height is 5 feet and whose base has a radius of 2 feet? How many square feet in the whole surface?

11. How many cubic feet in a right quadrangular pyramid whose altitude is 10 feet, and whose base is 3 feet square?

12. How many cubic feet in the cone whose dimensions are given in the tenth example?

13. The slant height of a frustum of a right pyramid is 6 feet, and the perimeters of the two bases are 18 feet and 12 feet respectively; what is the convex surface of the frustum?

14. What would be the slant height of the pyramid whose frustum is given in the preceding example?

15. What is the whole surface of a frustum of a cone whose altitude is 8 feet, and of whose bases the radii are 11 feet and 5 feet respectively?

16. The altitude of a pyramid is 25 feet, and its base is a rectangle 8 feet by 6; how many cubic feet in the pyramid?

17. The altitude of a cone is 20 feet, and the radius of its base 5 feet; how many cubic feet in the cone?

18. How many cubic feet in a frustum of the cone given in the preceding example, cut off by a plane 5 feet from the base?

19. How far from the base must a cone whose altitude is 12 feet be cut off so that the frustum shall be equivalent to one half of the cone?

20. How many cubic metres in the frustum of a cone whose altitude is 20 metres, radius of lower base 10 metres, upper 6 metres?

21. How many cubic decimetres in a truncated triangular prism whose base is an equilateral triangle with a perimeter of 3 decimetres, and whose edges are 5, 7, and 8 decimetres, respectively?

22. What is the ratio of the lateral surfaces of a right circular cylinder and a right circular cone of the same base and altitude, if the altitude is three times the radius of the base?

23. In example 15, page 70, how many times as much paint would it take to cover the church as to cover the model, if the thickness of the coats of paint varied in the same ratio as the linear dimensions of the church and the model?

24. How many square metres in the entire surface of a regular tetraedron whose edge is one metre?

25. How many cubic metres in a regular tetraedron whose edge is one metre?

26. What is the length of the edge of a regular tetraedron whose volume is 50 cubic metres?

27. How many cubic decimetres in a regular octaedron whose edge is one decimetre?

28. How many square feet in the surface of a sphere whose radius is 6 feet?

29. How many cubic feet in a sphere whose radius is 8 feet?

30. What is the ratio of the volumes of two spheres whose radii are as 4 : 8?

31. What is the radius of a sphere whose volume is 100 cubic metres?

32. Are spheres always similar solids? Are cones?

33. What is the least number of planes that can enclose a space?

EXERCISES.

100. The convex surfaces of right prisms of equal altitudes are as the perimeters of their bases. (21.)

101. The four diagonals of a parallelopiped bisect each other.

102. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges. (II. 74.)

103. In a rectangular parallelopiped the diagonals are equal; and the square of each is equal to the sum of the squares of the three dimensions.

104. In a cube the square of a diagonal is three times the square of an edge.

105. Polygons formed by parallel planes cutting a pyramid are as the squares of their distances from the vertex. (57; II. 79.)

106. The volume of a triangular prism is equal to half the product of the area of a lateral face and the perpendicular distance of this face from the opposite edge.

107. Of prisms of equal bases and equal altitudes, the right prism has the minimum surface; and of right prisms of equivalent bases and equal altitudes that whose base is a regular polygon has the minimum surface.

108. Of all quadrangular prisms of equivalent volumes the cube has the minimum surface.

109. The lateral surface of a pyramid is greater than the base.

110. Tetraedrons are equal, if the parts are similarly situated, —

1st. When three faces of the one are respectively equal to three faces of the other.

2d. When the six edges of the one are respectively equal to the six edges of the other.

3d. When two faces and the included diedral angle of the one are respectively equal to two faces and the included diedral angle of the other.

111. Tetraedrons are similar, if the parts are similarly situated,—

1st. When three faces of the one are respectively similar to three faces of the other.

2d. When the homologous edges have a constant ratio.

3d. When two triedral angles of the one are equal to two triedral angles of the other.

4th. When a diedral angle of the one is equal to a diedral angle of the other, and the faces including these angles are respectively similar.

112. Two tetraedrons having a triedral angle of the one equal to a triedral angle of the other are to each other as the products of the edges of the equal triedral angles. (70; II. 116, 55.)

113. State and prove the converse of Theorem XXIII.

114. How can Theorem XII. be proved from Theorem XIII.?

115. If a pyramid is cut by a plane parallel to its base, the pyramid cut off will be similar to the whole pyramid. (57; 79.)

116. In a sphere great circles bisect each other.

117. A great circle bisects a sphere. (88.)

118. The centre of a small circle is in the perpendicular from the centre of the sphere to the small circle.

119. Small circles equally distant from the centre of a sphere are equal.

120. The intersection of two spheres is a circle.

121. The arc of a great circle can be made to pass through any two points on the surface of a sphere. (VI. 2.)

122. *Definition.* A plane is tangent to a sphere when it touches but does not cut the sphere.

123. Prove that the radius of a sphere to the point of tangency of a plane is perpendicular to the plane. (VI. 9.)

(For Fig. for 124–127 see page 188.)

124. As the semi-decagon revolves about $A\,F$, what kind of a solid is described by $A\,B$? What by $B\,C$? By $C\,D$?

125. The convex surface described by $A\,B = A\,K \times \text{circ. } G\,O$.

Draw from G a perpendicular to $A\ B$, and from the point where it meets $A\ B$ a perpendicular to $A\ F$. (60.)

126. The convex surface described by $C\ D = L\ M \times$ circ. $G\ O$. (22.)

127. *Definition.* The surfaces described by $A\ B$, $B\ C$, $C\ D$, &c., are called *zones.*

128. The area of a zone is equal to the product of its altitude by the circumference of a great circle.

129. Zones on the same or equal spheres are as their altitudes.

130. The surface of a sphere is four times the surface of one of its great circles. (96 ; III. 50.)

131. *Definition.* A polyedron is circumscribed about a sphere when its faces are each tangents to the sphere. In this case the sphere is inscribed in the polyedron.

132. The surface of a sphere is equal to the convex surface of the circumscribed cylinder. (96 ; 22.)

133. *Definition.* A **Spherical Sector** is the solid described by any sector of a semicircle as the semicircle revolves about its diameter.

134. The volume of a spherical sector is equal to the product of the surface of the zone forming its base by one third of the radius of the sphere of which it is a part.

135. *Definition.* A **Spherical Segment** is a part of a sphere included by two parallel planes cutting or touching the sphere. When one plane touches and one cuts the sphere, the spherical segment is called *a spherical segment of one base;* when both cut, *a spherical segment of two bases.*

136. How can the volume of a spherical segment of one base be found ? A spherical segment of two bases ?

137. A sphere is two thirds of the circumscribed cylinder.

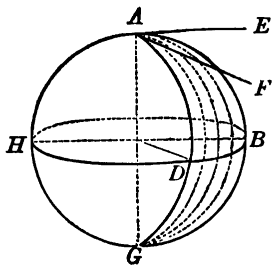
138. A cone, hemisphere, and cylinder having equal bases and the same altitude are as the numbers 1, 2, 3.

BOOK VIII.

SPHERICAL GEOMETRY.

DEFINITIONS.

1. Spherical Geometry treats of spherical magnitudes; particularly of figures formed on the surface of a sphere by arcs of circles of the sphere.



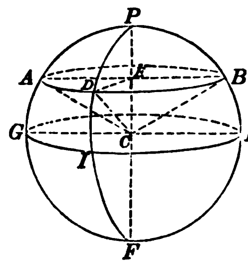
2. When on the surface of a sphere two arcs of a great circle intersect, their difference of direction at the point of intersection is called the *spherical angle* of the arcs. Thus, the spherical angle of the arcs $A\,B$, $A\,D$, is their difference of direction at A.

3. *Corollary.* As the direction of a curve at any point is the same as the direction of its tangent at that point, the spherical angle of two intersecting arcs is the same as the angle of the tangents to the arcs at the point of intersection. Thus, if $A\,E$ in the plane $A\,B\,G$, and $A\,F$ in the plane $A\,D\,G$, are respectively tangents to the arcs $A\,B$ and $A\,D$ at A, the spherical angle of the arcs $A\,B$, $A\,D$ is the same as the angle $E\,A\,F$.

4. The **Poles** of a circle of a sphere are the extremities of that diameter of the sphere which is perpendicular to the plane of the circle. This diameter is called the *axis* of the circle. Thus, the diameter of the sphere $A\,G$, perpendicular to the plane $B\,D\,H$, is the axis, and A and G the poles of the circle $B\,D\,H$.

THEOREM I.

5. *All the points in the circumference of a circle of a sphere are equally distant from its poles.*



Let P, the extremity of the diameter $P F$, be the pole of the circle $A B D$; the points A, B, D are equally distant from P.

For E, the point of intersection of $P F$ with the plane of $A B D$, is the centre of the circle $A B D$ (VII. 92), and $E A$, $E B$, $E D$ are equal; hence the distances $P A$, $P B$, $P D$ are equal (VI. 7); and therefore the arcs of great circles $P A$, $P B$, $P D$ are equal (III. 12).

6. *Scholium.* The distance between two points on the surface of a sphere is the length of the arc of a great circle drawn between the points. (See 17.)

7. *Cor.* 1. If $G H I$ is a great circle, the polar distance $P G$ is quarter of a circumference, or a quadrant; for $P G$ is the measure of the right angle $P C G$, whose vertex is at the centre (III. 17).

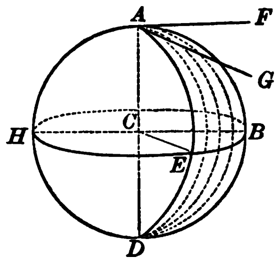
8. *Cor.* 2. A point on the surface of a sphere a quadrant's distance from two other points is the pole of a great circle passing through these two points. For (Fig. in 10), $A B$ and $A E$ being quadrants, $A C B$ and $A C E$ are right angles, therefore $C A$ is perpendicular to the plane of the arc $B E$ (VI. 10); and P is the pole of the arc $B E$ (4).

9. *Cor.* 3. By means of poles arcs may be drawn on the surface of a sphere. Thus, if the arc $P A$ revolves about P as a centre, the point A will describe the circumference of the small circle $A B D$; and if the quadrant $P G$ revolves about P as a centre, the point G will describe the circumference of the great circle $G H I$.

THEOREM II.

10. *The spherical angle of two arcs of great circles is equal to the diedral angle of their planes, and is measured by the arc of a great circle described from the vertex as a pole and included between its sides.*

Let C be the centre of the sphere, and $A\,B$, $A\,E$ the arcs of two great circles intersecting at A.



Draw $A\,F$, $A\,G$ tangents respectively to the circumferences $A\,B\,D$, $A\,E\,D$. The spherical angle of the arcs is $F\,A\,G$ (3); and the line of intersection, $A\,D$, of the planes of the arcs $A\,B$, $A\,E$, is a diameter of the sphere (VI. 3; VII. 94).

1st. As $A\,F$ in the plane $A\,B\,C$, and $A\,G$ in the plane $A\,E\,C$, are each perpendicular to $A\,C$, the edge of the diedral angle of the planes $A\,B\,L$, $A\,E\,C$, the angle $F\,A\,G$ is the measure of the diedral angles of these planes (VI. 34); therefore the spherical angle of the arcs is equal to the diedral angle of the planes.

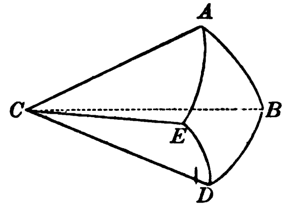
2d. Let $B\,E$ be the arc of a great circle described from A as a pole, and intersecting the arcs $A\,B$, $A\,E$ (produced if necessary) in B and E, and let the plane of the arc $B\,E$ intersect the planes $A\,B\,C$, $A\,E\,C$, in $B\,C$ and $E\,C$. $A\,C$ is at right angles to the plane $B\,E\,C$ (4); therefore $B\,C$, $E\,C$ are perpendicular to $A\,C$ (VI. 5), and the angle $B\,C\,E$ is the measure of the diedral angle of the planes $A\,B\,C$, $A\,E\,C$ (VI. 34). But $B\,E$ is the measure of the angle $B\,C\,E$ (III. 16); hence $B\,E$ is also the measure of the spherical angle of the arcs $A\,B$, $A\,E$.

11. *Corollary.* A great circle whose circumference passes through the pole of another great circle is perpendicular to that circle; and the poles of the first circle are in the circumference of the second.

Conversely. If two great circles are perpendicular to each other, the circumference of each must pass through the poles of the other.

SPHERICAL POLYGONS.

DEFINITIONS.



12. A **Spherical Polygon** is a portion of the surface of a sphere included by three or more arcs of great circles, each of these arcs being less than a semi-circumference; as $A\ B\ D\ E$.

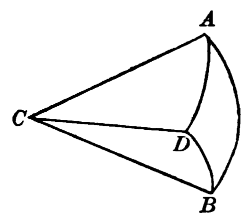
These arcs are called the *sides* of the polygon, and the spherical angles of the arcs the *angles* of the polygon.

13. A **Spherical Triangle** is a spherical polygon of three sides. The terms *right-angled, isosceles, equilateral,* &c., are used with the same signification as in plane triangles.

14. The polyedral angle C formed at the centre of the sphere, and included by the planes of the arcs of a spherical polygon, is called the *corresponding polyedral angle* of the spherical polygon.

THEOREM III.

15. *Any side of a spherical triangle is less than the sum of the other two.*



Let $A\ B\ D$ be a spherical triangle; then $A\ B < A\ D + D\ B$.

Let C be the corresponding triedral angle of the triangle $A\ B\ D$. Now the face angle (VI. 59)

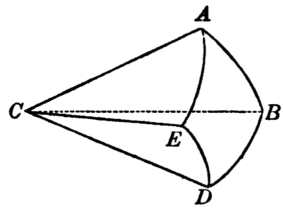
$$A\,C\,B < A\,C\,D + D\,C\,B$$

But $A\ B$ is the measure of the angle $A\ C\ B$, $A\ D$ of $A\ C\ D$, and $D\ B$ of $D\ C\ B$; hence

$$A\,B < A\,D + D\,B.$$

THEOREM IV.

16. *The sum of the sides of a spherical polygon is less than the circumference of a great circle.*

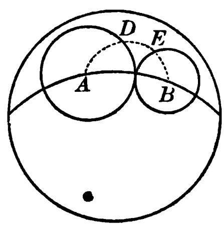


Let $A\,B\,D\,E$ be a spherical polygon; then $A\,B + B\,D + D\,E + E\,A <$ circ. of a great circle.

Let C be the corresponping polyedral angle of the polygon. Now the face angle $A\,C\,B + B\,C\,D + D\,C\,E + E\,C\,A < 4$ right angles (VI. 60). But the sides of the spherical polygon are respectively the measures of these angles. Therefore $A\,B + B\,D + D\,E + E\,A < 4$ right angles, or 4 quadrants, that is, less than a great circle.

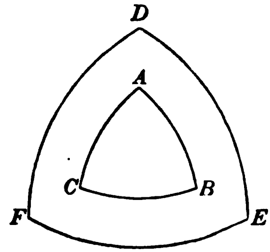
THEOREM V.

17. *The shortest distance on the surface of a sphere from one point to another is the arc of a great circle joining the points.*



Let $A\,B$ be the arc of a great circle drawn between A and B; the arc $A\,B$ is shorter than any other line from A to B on the surface of the sphere. Take any point C of the arc $A\,B$, and with A and B respectively as poles, and the polar distances $A\,C$ and $B\,C$, describe circumferences on the surface of the sphere. These circumferences touch at C, but lie wholly without each other. For let D be any point in the circumference whose pole is A; and from D draw arcs of great circles to A and B respectively. $A\,D + D\,B > A\,B$ (15); subtracting from each side $A\,D = A\,C$, we have $B\,D > B\,C$; therefore D is without the circumference whose pole is B. Now the distance from A to D cannot be less than the distance from A to C (5), nor from B to E less than from B to C. Therefore the shortest distance from A to B must pass through C; and C is any point in the arc

of a great circle drawn from A to B. Therefore the shortest distance between two points on the surface of a sphere is the arc of a great circle joining the points.

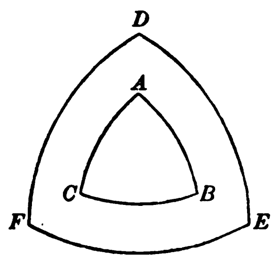


18. *Definition.* If from the vertices of a spherical triangle as poles arcs of great circles are described, these arcs intersecting form a triangle which is called the *polar triangle* of the first triangle.

Thus if EF, FD, DE are the arcs of great circles whose poles are respectively A, B, C, the vertices of the spherical triangle ABC, DEF is the polar triangle of ABC. As all great circles intersect in two points, four triangles can be formed by the intersection of the arcs described from A, B, C, as poles; but that one whose vertex D, homologous to A, is on the same side of BC as A, is the polar triangle.

THEOREM VI.

19. *A spherical triangle is itself the polar triangle of its polar triangle.*

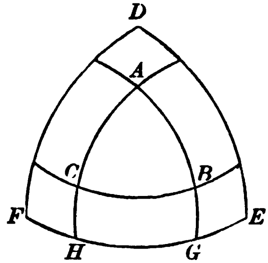


Let DEF be the polar triangle of the spherical triangle ABC; then, reciprocally, ABC is the polar triangle of DEF.

For as A is the pole of the arc EF, F is a quadrant's distance from A; and as B is the pole of the arc FD, F is a quadrant's distance from B; therefore two points of the arc AB being a quadrant's distance from F, F on the same side of AB as its homologous angle C, must be the pole of the arc AB (8). In like manner it can be shown that D is the pole of the arc BC, and E of the arc CA. Therefore ABC is the polar triangle of DEF.

THEOREM VII.

20. *The sides of a spherical triangle are the supplements respectively of the measures of the angles of its polar triangle.*



Let $A\,B\,C$ be the polar triangle of $D\,E\,F$; then $F\,E$ is the supplement of the measure of the angle A, $F\,D$ of B, $D\,E$ of C. Produce if necessary the sides $A\,B$, $A\,C$ till they meet $E\,F$ in G and H. As A is the pole of the arc $H\,G$, $H\,G$ is the measure of the angle A (10). As F is the pole of the arc $A\,G$, $F\,G$ is a quadrant (7); and as E is the pole of $A\,H$, $E\,H$ is also a quadrant; hence

$$FG + HE = FE + HG = 2 \text{ quadrants or a semi-circumference;}$$

that is, $F\,E$ is the supplement of $H\,G$, the measure of the angle A.

In like manner it can be shown that $D\,F$ and $D\,E$ are respectively the supplements of the measures of the angles B and C.

THEOREM VIII.

21. *On the same, or equal spheres, triangles mutually equilateral are also mutually equiangular.*

For as the sides of the triangles are the measures respectively of the face angles of their corresponding triedral angles (10), their face angles are respectively equal; hence their corresponding triedral angles are equal or symmetrical (VI. 62, 63), and therefore the diedral angles of the faces of these triedral angles are equal; that is, the angles of the triangles are respectively equal (10).

22. *Scholium.* In equal spherical triangles the equal angles are opposite the equal sides.

THEOREM IX.

23. *On the same, or equal spheres, triangles mutually equiangular are also mutually equilateral.*

For their polar triangles will be mutually equilateral (20); therefore their polar triangles will be mutually equiangular (21); hence also the polar triangles of the polar triangles, that is (19) the triangles themselves, will be mutually equilateral (20).

24. *Scholium.* Plane triangles that are mutually equiangular are similar (II. 55), but not necessarily equilateral. Spherical triangles which are mutually equiangular are also equilateral only when they are on the same or equal spheres; on unequal spheres mutually equiangular spherical triangles will not be equal, but similar, and their homologous sides will be as the radii of the spheres.

THEOREM X.

25. *On the same, or equal spheres, triangles having two sides and the included angle, or two angles and the included side respectively equal, when their sides are similarly situated, are equal.*

For as in plane triangles (I. 80, 81), one of these triangles can be placed upon the other so as exactly to coincide.

26. *Cor.* 1. As on the same, or equal spheres, two triangles mutually equilateral are also mutually equiangular (21), and triangles mutually equiangular are also mutually equilateral (23); therefore on the same, or equal spheres, triangles mutually equilateral, or mutually equiangular, when their sides are similarly situated, are equal.

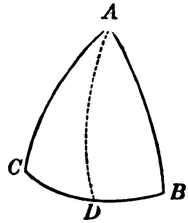
27. *Scholium.* In all these cases, if the sides are not similarly situated, the triangles cannot be made to coincide. On the same, or equal spheres, triangles mutually equal in their parts, but having their sides not similarly situated, are called *symmetrical* with one another. (See 34.)

28. *Cor.* 2. On the same, or equal spheres, isosceles triangles having two sides and the included angle, or two angles and the included side, respectively equal, or being mutually equilateral, or mutually equiangular, can always be made to coincide, and are therefore equal.

THEOREM XI.

29. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*

In the isosceles spherical triangle $A\,B\,C$ let $A\,C$ and $A\,B$ be the equal sides; then the angle B is equal to the angle C. Draw $A\,D$ the arc of a great circle to D, the middle of the base $C\,B$. The triangles $A\,D\,C$, $A\,B\,D$ are mutually equilateral, and therefore mutually equiangular (21); hence the angle $B = C$.



30. *Cor.* 1. In an isosceles spherical triangle, therefore, an arc of a great circle fulfilling any one of the following conditions fulfils them all:

1. Bisecting the vertical angle.
2. Bisecting the base at right angles.
3. Drawn from the vertex bisecting the base.
4. Drawn from the vertex perpendicular to the base.
5. Drawn from the vertex bisecting the triangle.
6. Bisecting the triangle and the base.
7. Bisecting the triangle and perpendicular to the base.

31. *Cor.* 2. An equilateral spherical triangle is equiangular.

THEOREM XII.

32. *If two angles of a spherical triangle are equal, the sides opposite are also equal.*

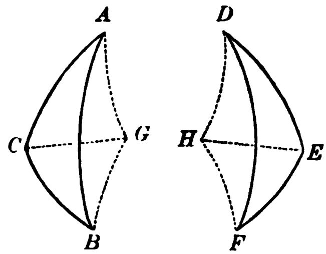
For in this case the corresponding sides of the polar triangle are equal (20); hence the angles opposite the equal sides of

the polar triangle are also equal (29); and therefore the corresponding sides of the polar triangle of the polar triangle, that is (19), the corresponding sides of the given triangle are equal.

33. *Corollary.* An equiangular spherical triangle is equilateral.

THEOREM XIII.

34. *Symmetrical spherical triangles are equivalent.*



Let $A B C$, $D E F$ be two symmetrical spherical triangles; $A B C$ is equivalent to $D E F$. Let G be the pole of the small circle passing through the three points A, B, C. Draw $G A$, $G B$, $G C$, arcs of great circles; then (5) $G A = G B = G C$; hence (29) angle $G A C = G C A$, and $G C B = G B C$. From the points D, E, F draw arcs of great circles so that angle $E D H = C A G$, $D E H = A C G$, and $E F H = C B G$. As angle $G A C = G C A$, $G A C = E D H$, $G C A = H E D$, therefore $H D E = H E D$, and $D H = E H$ (32). As the whole angle $C = E$, and $A C G = D E H$, therefore $H E F = G C B$. But as $G C B = G B C$, and $G B C = H F E$, therefore $H F E = H E F$, and $F H = E H$ (32).

Now the triangles $G A C$, $H D E$, being isosceles, and having $A C$ and its adjacent angles equal respectively to $D E$ and its adjacent angles, are equal (28); likewise the triangles $G B C$, $H E F$, being isosceles, and having $C B$ and its adjacent angles equal respectively to $E F$ and its adjacent angles, are equal. Therefore in both triangles $H D E$, $H E F$, the side $E H = G C$, that is, $F H$ and $D H$ meet $E H$ at the same point H.

Now as $H D = H E = H F$, $H D F$ is an isosceles triangle; $A G B$ is also isosceles; and as the angle $A G C = D H E$, and

$C\,G\,B = E\,H\,F$, the whole angle $A\,G\,B = D\,H\,F$; and the triangle $A\,G\,B = D\,F\,H$ (28).

Now as the triangle		$A\,G\,C = D\,E\,H$
and "	"	$C\,G\,B = H\,E\,F$
" "	"	$A\,G\,B = D\,F\,H$

therefore $A\,G\,C + C\,G\,B - A\,G\,B = D\,E\,H + H\,E\,F - D\,F\,H$

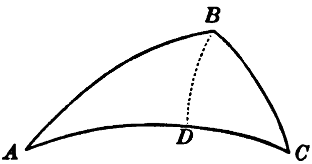
or $A\,B\,C = D\,E\,F$

85. *Scholium.* The poles G and H might lie within $A\,B\,C$ and $D\,E\,F$; in this case the triangles $A\,G\,B$, $D\,F\,H$ must be added to make up the triangles $A\,B\,C$, $D\,E\,F$.

THEOREM XIV.

86. *In a spherical triangle the greater side is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.*

In the spherical triangle $A\,B\,C$ let the angle $A\,B\,C > C$; then $A\,C > A\,B$



Draw the arc of a great circle $B\,D$, making the angle $C\,B\,D = B\,C\,D$; then (32),

$$D\,B = D\,C$$

and $A\,C = A\,D + D\,C = A\,D + D\,B$

But (15) $A\,D + D\,B > A\,B$

therefore $A\,C > A\,B$

Conversely. Let $A\,C > A\,B$; then the angle $A\,B\,C > C$. For if the angle $A\,B\,C$ is not greater than C, it must be either equal to it or less. It cannot be equal to it, because then $A\,C = A\,B$ (32), which is contrary to the hypothesis. It cannot be less, because then by the former part of this theorem $A\,C < A\,B$, which is also contrary to the hypothesis. Hence the angle $A\,B\,C > C$.

THEOREM XV.

37. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

The sum of the arcs which measure the angles of a spherical triangle together with the three sides of its polar triangle is equal to three semi-circumferences (20). But the three sides of the polar triangle are together less than two semi-circumferences (16); hence the sum of the arcs which measure the angles of the given triangle alone must be greater than one semi-circumference; therefore the sum of the angles is greater than two right angles.

And as each angle is less than two right angles, the sum of the three angles must be less than six right angles.

38. *Corollary.* A spherical triangle may have two, or even three right angles; or two, or even three obtuse angles.

If a spherical triangle has three right angles its sides are each quadrants, and the triangle is called a *tri-rectangular* triangle; each of its sides is a quadrant, each vertex is the pole of the opposite side, and its polar triangle coincides with it, that is, it is its own polar triangle. A tri-rectangular spherical triangle is evidently an eighth part of the surface of the sphere.

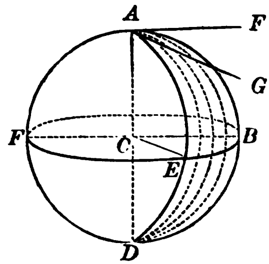
DEFINITIONS.

39. A **Lune** is a portion of the surface of a sphere included between two semi-circumferences of great circles, as $A\,B\,D\,E\,A$ in (41).

40. A **Spherical Ungula**, or **Wedge**, is a portion of a sphere included between the halves of two great circles and the lune formed by the semi-circumferences of these circles; as the solid whose faces are the planes $A\,B\,D$, $A\,E\,D$, and the lune $A\,B\,D\,E\,A$.

THEOREM XVI.

41. *A lune is to the surface of the sphere as the angle of the lune is to four right angles.*



Let $A\,B\,D\,E\,A$ be a lune on the sphere whose centre is C, and let A and D be the poles of the great circle $B\,E\,F$; then the lune $A\,B\,D\,E\,A$ is to the surface of the sphere as the angle at A, or $B\,C\,E$, is to four right angles, that is, as the arc $B\,E$ is to the circumference of the sphere. Suppose the circumference $B\,E\,F$ to be divided into any number of equal parts, of which the arc $B\,E$ contains an exact number, and planes be passed through the points of division and the diameter $A\,D$. The whole surface of the sphere will be divided into equal lunes (as they can be made to coincide with each other) of which the given lune will evidently contain as many as there are parts of the arc $B\,E$. Hence the lune $A\,B\,D\,E\,A$ is to the surface of the sphere as the arc $B\,E$ is to the circumference $B\,E\,F$; that is, as the angle of the lune is to four right angles.

42. *Scholium.* If the circumference $B\,E\,F$ and the arc $B\,E$ are incommensurable, the proposition is proved by the same method as that used in (II. 35).

43. *Cor.* 1. On the same or equal spheres lunes are to each other as their angles.

44. *Cor.* 2. If we let T represent the surface of the tri-rectangular spherical triangle, the surface of the sphere will be $8\,T$ (38); if we let L represent the lune, A its angle, and assume the right angle as unity, we have

$$L : 8\,T = A : 4$$

$$L = 2\,A \times T$$

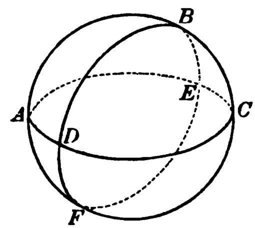
45. *Cor.* 3. The spherical ungula included by the planes ABD, AED is to the sphere as the arc BE is to the circumference of the sphere, that is, as the diedral angle of its planes is to four right angles. Taking the same notation as in (44), and letting V represent the sphere and U the ungula, we have

$$U : V = A :: 4$$

$$U = \frac{A}{4} \times V$$

THEOREM XVII.

46. *If two great circles intersect each other on the surface of a hemisphere, the sum of the vertical triangles thus formed is equivalent to a lune whose angle is equal to the angle of inclination of the two circles.*



Let the great circles ABC, DBE cut each other at B on the surface of the hemisphere above the great circle $AECD$: the sum of the triangles ABD, BEC is equal to the lune whose angle is ABD.

Produce the arcs ABC, DBE till they meet at F. The arcs ABC, BCF are each semicircumferences: taking away the common part BC, we have $AB = CF$. In like manner we can prove $DB = EF$, and $AD = CE$. Therefore the two triangles ABD, EFC being mutually equilateral are equivalent (28). Hence the triangle $ABD + BEC$ is equivalent to $EFC + BEC$, that is, to the lune $BEFCB$ whose angle is $EBC = ABD$.

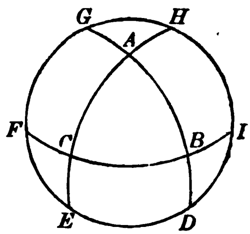
47. *Corollary.* By (44), triangle $ABD + BEC = 2B \times T$.

THEOREM XVIII.

48. *If a right angle is taken as unity, the area of a spherical triangle is equal to the sum of its angles minus two right angles multiplied by the tri-rectangular triangle.*

Let $A\,B\,C$ be a spherical triangle;

then $ABC=(A+B+C-2)\times T$



Produce the sides of $A\,B\,C$ till they meet the circumference of a great circle $G\,I\,E$ enclosing the triangle. By (46) we have

$$\begin{array}{ll}\text{triangle} & A\,D\,E+A\,G\,H=2\,A\times T\\ \text{"} & B\,F\,G+B\,I\,D=2\,B\times T\\ \text{"} & C\,H\,I+C\,E\,F=2\,C\times T\end{array}$$

The sum of these triangles is equal to the surface of the hemisphere, GIE, or $4\,T$, plus twice the triangle ABC; that is,

$$4\,T+2\,A\,B\,C=2\,(A+B+C)\times T$$

or $$2\,A\,B\,C=2\,(A+B+C)\times T-4\,T$$

whence $$A\,B\,C=(A+B+C-2)\times T$$

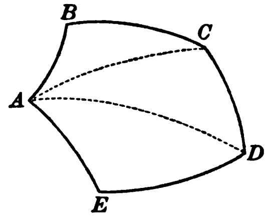
49. *Scholium.* It must be remembered that the angles are to be expressed in terms whose unit is the right angle. For example, if A is two thirds of a right angle, B a right angle, and C a right angle and a half,

$$A\,B\,C=(\tfrac{2}{3}+1+\tfrac{3}{2}-2)\,T=\tfrac{7}{6}\,T$$

that is, the area of $A\,B\,C$ is seven sixths of the area of the tri-rectangular triangle.

THEOREM XIX.

50. *The right angle being taken as unity, the area of a spherical polygon is equal to the sum of its angles minus as many times two right angles as it has sides less two, multiplied by the tri-rectangular triangle.*



Let $ABCDE$ be a spherical polygon; then if n represents the number of its sides, and S the sum of its angles,

$$ABCDE = \{S - (n-2)\,2\} \times T$$

For if from any vertex, A, arcs of great circles be drawn to the other vertices, the polygon will be divided into as many spherical triangles as it has sides less two; and the surface of each triangle (48) is equal to the sum of its angles minus two right angles multiplied by the tri-rectangular triangle. Now the sum of the triangles is the polygon itself, and the sum of the angles of the triangles the sum of the angles of the polygon; therefore

$$ABCDE = \{S - (n-2)\,2\} \times T$$

51. It is impossible on plane surfaces to present clearly the figures of Solid and Spherical Geometry. In teaching this part of Geometry the teacher should be supplied with a full set of geometric models. A common geographical globe will serve to illustrate many of the propositions of Spherical Geometry. The equator and meridians are great circles; the parallels of latitude, the tropics, and polar circles are small circles whose poles are the poles of the earth. In Navigation Theorem V. is a very important proposition. In Spherical Geometry the figures can be best presented on a blackboard which has the form of a sphere.

PRACTICAL QUESTIONS.

1. If the angles of a spherical triangle are 51°, 73°, and 97°, how many degrees are there in each side of its polar triangle?

2. How many square metres in a lune whose arcs make an angle of 85°, on a sphere whose radius is 5 metres?

3. How many square metres in a spherical triangle whose angles are 147°, 163°, and 105°, on a sphere whose diameter is 1.6 metres?

4. How many cubic metres in the spherical ungula whose spherical surface is the lune mentioned in Ex. 2?

5. How many cubic metres in a spherical polygon whose angles are 137°, 175°, 164°, 106°, 175°, and 141°, on a sphere whose radius is 3 metres?

6. How many cubic metres in the spherical pyramid whose base is the spherical polygon mentioned in Ex. 5?

EXERCISES.

52. Many of the propositions of Book VI. can be proved from the principles of Book VIII., and *vice versa*, and are essentially the same things.

Compare (VI. 30, 31) with (2, 3); (VI. 33) with (10); (VI. 57, 58) with (12 – 14); (VI. 59) with (15); (VI. 60) with (16); (VI. 77, 78) with (18 – 20); (VI. 62) with (21); (VI. 68) with (23 – 26); (VI. 63) with (27); (VI. 70) with (29 – 36); (VI. 79) with (37, 38).

53. In the same manner as Theorem XIII. (34) is proved, prove that triedral angles whose face angles are respectively equal, but not similarly situated (VI. 63), are equivalent, or symmetrically equal.

54. If the sum of the sides of a spherical polygon become equal to the circumference of a great circle (16), what then? Compare (VI. 67).

55. Show that two spherical triangles may have three parts of which one is a side, or even four of which two are sides, respectively equal, and be neither equal nor symmetrical. Compare (VI. 71).

56. State and prove as propositions of Spherical Geometry 103, 104, 105, 106, 107, 108 of Book I. Compare these with 72, 73, 74, 75 of Book VI.

57. If one side of a spherical triangle is a quadrant, an adjacent angle is acute, or right, or obtuse, according as the side opposite is less than, equal to, or greater than a quadrant. Compare (VI. 76).

58. If the sum of the angles of a spherical triangle become equal to two right angles (37), what then? what, if equal to six? Compare (VI. 80).

59. A spherical polygon is always within the surface of a hemisphere.

60. *Scholium.* It must be understood that the spherical polygon is convex, that is, that none of its sides produced would cut the polygon (VI. 61). If the spherical polygon were not convex some of its angles would exceed two right angles, and the sum of its angles would be unlimited.

61. If the sides of a spherical triangle are produced there will be formed on the surface of the sphere seven other spherical triangles. Compare these triangles with each other and with the original triangle.

62. If the planes of the corresponding polyedral angle of a spherical polygon are produced, their intersections with the opposite surface of the sphere will form a spherical polygon symmetrically equal to the given spherical polygon.

63. If at the vertex of the corresponding triedral angle of a spherical triangle perpendiculars are erected to the several faces, these perpendiculars will pass through the vertices of the polar triangle of the given triangle (VI. 77).

64. If S represents the sum of the angles and n the number of the sides of a spherical polygon,

then $$S > (n-2)\,2 \text{ but } < (n-2)\,6$$

BOOK IX.

LOCI.

DEFINITIONS.

1. A **Geometric Locus** is the position of all the points that have a common property.

2. The **Locus of a Point** is the line described by that point when moving according to some fixed law. The moving point is called the *generatrix* of the line.

3. The **Locus of all the Points** in space may be a line or a surface.

THEOREM I.

4. *The locus of a point equally distant from the extremities of a straight line is a perpendicular bisecting this line.*

For demonstration see (I. 94).

5. *Corollary.* The locus of all the points equally distant from the extremities of a line is a plane bisecting the line and perpendicular to it.

THEOREM II.

6. *The locus of a point equally distant from the sides of an angle is a line bisecting this angle.*

For demonstration see (I. 103).

7. *Corollary.* The locus of all the points equally distant from the sides of an angle is a plane bisecting the angle and perpendicular to the plane of its sides.

THEOREM III.

8. *The locus of a point at a given distance from a given point is the circumference of a circle described from the given point as a centre with the given distance as a radius.*

See (III. 1).

9. *Corollary.* The locus of all the points at a given distance from a given point is the surface of a sphere with the given point as a centre and the given distance as a radius. (VII. 88.)

THEOREM IV.

10. *The locus of the middle points of parallel chords of a circle is the diameter perpendicular to these chords.*

For demonstration see (III. 13).

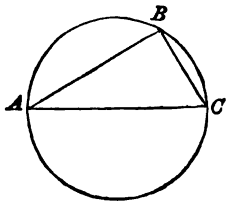
THEOREM V.

11. *The locus of the middle points of equal chords of a circle is the circumference of a circle concentric with the given circle and having a radius equal to the distance of these chords from the centre.*

See (III. 67).

THEOREM VI.

12. *The locus of a point such that the sum of the squares of its distances from two given points is equal to the square of the distance between the points is the circumference of a circle whose diameter is the distance between the points.*



Let A and C be the given points, and B any point in the circumference of a circle described on $A\,C$ as a diameter. Draw $B\,A$ and $B\,C$. B is a right angle (III. 25); therefore (II. 66)

$$\overline{A\,B}^2 + \overline{B\,C}^2 = \overline{A\,C}^2$$

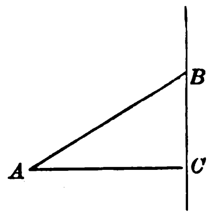
13. *Corollary.* The locus of all the points such that the sum of the squares of the distances from two given points is

equal to the square of the distance between the points is the surface of a sphere whose diameter is the distance between the points.

THEOREM VII.

14. *The locus of a point such that the difference of the squares of its distances from two given points is equal to the square of the distance between the points is a line through either of the points perpendicular to the line joining the given points.*

Let A and C be the given points, and B any point in $B\,C$ perpendicular to $A\,C$ at the point C. Then (II. 67)



$$\overline{A\,B}^2 - \overline{B\,C}^2 = \overline{A\,C}^2$$

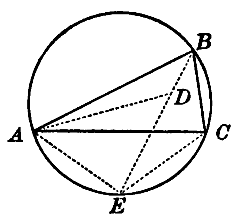
15. *Corollary.* The locus of all the points such that the difference of the squares of the distances from two given points is equal to the square of the distance between the points is a plane passing through either of the points and perpendicular to the line joining the given points.

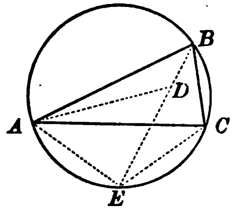
PROPOSITION VII.

PROBLEM.

16. *To find the locus of the centre of a circle inscribed in a triangle whose base and vertical angle are given.*

With $A\,C$, the given base, as a chord, describe a circle whose segment $A\,B\,C$ shall contain an angle equal to the given angle (V. 19). Draw $A\,B$ and $B\,C$. $A\,B\,C$ is one of the triangles proposed. Draw $A\,D$ bisecting the angle $C\,A\,B$, and $B\,D$ bisecting $A\,B\,C$; their point of intersection D will be the centre of the circle inscribed in $A\,B\,C$ (V. 17). Produce $B\,D$ to meet the circumference $A\,B\,C$ in E; draw $E\,A$ and $E\,C$.





The angle (III. 24)

$$EAC = EBC = ABD$$

and $CAD = DAB$

Hence $EAD = ABD + DAB$

but (I. 79) $ABD + DAB = ADE$

Therefore, $EAD = ADE$

and (I. 85) $EA = ED$

In like manner $EC = ED$, and the arc whose centre is E and radius ED is the locus required.

17. *Corollary.* If the given angle is a right angle the locus is a quadrant, if an acute angle greater than a quadrant, if obtuse less.

EXERCISES.

18. What is the locus of all the points that are the same distance from a given line?

19. What is the locus of a point equally distant from three given points?

20. What is the locus of a point equally distant from three given planes?

21. What is the locus of a point at a given distance from a given line?

22. What is the locus of a point at a given distance from two given parallel lines?

23. What is the locus of a point whose distances from two given straight lines are in a given ratio?

24. What is the locus of the centres of the circles having the same radius and tangent to the circumference of a given circle?

25. What is the locus of the middle points of all the straight lines that have one extremity in a given point and the other in a given straight line?

26. What is the locus of the vertices of all the equivalent triangles on the same base?

EXERCISES.

BOOK I.

160. BD bisects AC, the base of the triangle ABC; then

$$BD < \tfrac{1}{2}(AB + BC)$$

and

$$BD > \tfrac{1}{2}(AB + BC - CA)$$

161. The semi-perimeter of a triangle is greater than any side, and less than the sum of any two sides of the triangle.

162. *Definition.* — Of the two lines drawn from a vertex of a triangle to the opposite side, the one bisecting the angle is called the *bisector*, the one bisecting the opposite side the *medial line*.

163. A medial line in any triangle is less than half the sum of the sides including the angle from whose vertex the medial line is drawn.

164. The sum of the three medial lines of a triangle is less than the perimeter, but greater than the semi-perimeter of the triangle.

165. In a triangle ABC, if BD and AE are drawn perpendicular to a line through C, if F is the middle point of AB, then $FD = FE$.

166. In a triangle ABC, on AB, produced if necessary, take $AD = AC$, and on AC take $AE = AB$; draw DE cutting BC in F, and join AF; AF bisects the angle BAC.

167. The two straight lines AB, CD intersect at E; the lines AC, DB are drawn forming the triangles AEC, EDB; the angles C, B are respectively bisected by CF, BF which meet at F; then the angle $F = \frac{1}{2}(A + B)$.

168. Of the three lines from the vertex of a triangle the bisector lies between the perpendicular and the medial line.

169. The difference of the angles at the base of a triangle is double the angle included by the bisector and the perpendicular from the same vertex.

170. If one angle at the base of a triangle is double the other, the side opposite the less angle is equal to the sum or difference of the segments of the base made by a perpendicular to it from the opposite vertex, according as the perpendicular falls without or within the triangle.

171. The angle included by a line bisecting an angle at the base of a triangle and a line bisecting the exterior angle at the other extremity of the base is equal to half the vertical angle of the triangle.

172. A perpendicular from an extremity of the base of an isosceles triangle to the opposite side cuts off an angle equal to half the vertical angle of the isosceles triangle.

173. The extremities of the base of an isosceles triangle are equally distant from the opposite sides.

174. The sum of any two lines drawn from a point in the base of an isosceles triangle, and making equal angles with it, to the opposite sides is constant, and equal to a line drawn at the same angle to the base from either extremity of the base to the opposite side.

175. The sum of the distances of any point in the base of an isosceles triangle from the sides is constant, and equal to the perpendicular from either extremity of the base to the opposite side. What if the point is in the base produced?

176. The sum of the perpendiculars drawn from any point within an equilateral triangle to the sides is constant, and equal to the perpendicular from any vertex to the opposite side. What if the point is without the triangle?

177. The perimeter of a parallelogram formed by the lines drawn from any point of the base of an isosceles triangle parallel to the sides and the segments of the sides is constant.

178. The sum of the segments of the sides of an isosceles triangle intercepted between the vertex and a line bisected by the base is constant, and equal to the sum of the two equal sides of the triangle.

179. If B is the vertex of an isosceles triangle, and CB is produced to D so that $BD = BC$ and DA is drawn, then CAD is a right angle.

180. If a straight line DE is drawn perpendicular to the base of an isosceles triangle ABC, cutting BC in F and AB produced in E, BEF is also an isosceles triangle.

181. If in the base AC of an isosceles triangle ABC a point D is taken, CE cut off equal to CD, and DE produced to meet AB in F, then $3\,BEF$ is equal to two right angles $+ F$, or to four right angles $+ F$, according as DE produced meets AB above or below the base.

182. If the straight lines bisecting the angles at the base of a triangle are equal the triangle is isosceles, and the angle included by the bisecting lines is equal to an exterior angle at the base of the triangles.

183. In a triangle if the straight lines drawn from the extremities of the base, bisecting the sides, or making equal angles with the sides, are equal, the triangle is isosceles.

184. If one angle of a triangle is double, or triple another, or equal to the sum, or to half the difference of the other two, the triangle can be divided into two isosceles triangles, or an isosceles triangle can be added to it so as to form with it an isosceles triangle.

185. The straight lines bisecting the vertical angle of a triangle and the lines bisecting the two exterior angles at the base intersect at the same point.

186. The perpendiculars from the vertices of a triangle to the opposite sides respectively bisect the angles of the triangle formed by joining the feet of these perpendiculars.

187. In the triangle $A B C$ let $B > C$; on $A C$ take $A D = A B$ and join $B D$; then angle $B D A = \frac{1}{2} (B + C)$, and $D B C = \frac{1}{2} (B - C)$.

188. The three lines drawn through the point of intersection of the bisectors of the angles of an equilateral triangle, parallel to the sides of the triangle, trisect each of these sides.

189. The angle included between the perpendicular to the hypothenuse from the vertex of the right angle of a right triangle and the medial line from the same vertex is equal to the difference of the acute angles of the right triangle.

190. The bisector of the right angle of a right triangle bisects the angle included between the medial line and the perpendicular to the hypothenuse from the same vertex.

191. $A B C$ is an equilateral triangle, $B D$ is perpendicular to $A C$, and $D E$ perpendicular to $B C$; then $E C = \frac{1}{3} B E$.

192. $A B C$ is a triangle right-angled at C, and angle $B = 2 A$; then $2 B C > A C$.

193. If one angle of a triangle is equal to the sum of the other two, the greatest side is equal to twice the medial line from the vertex of the greatest angle.

194. In a triangle an angle is acute, right, or obtuse, according as the medial line from its vertex is greater than, equal to, or less than, half the opposite side.

195. In the triangle $A B C$ if the angle C is bisected by a line meeting $A B$ in D, and through D a line parallel to $A C$ is drawn cutting $B C$ in E and the line bisecting the exterior angle at C in F, then $D E = E F$.

196. If the medial lines from the vertices of two angles of a triangle are produced until the parts without are equal to the parts within the triangle, respectively, the line joining the external extremities of these lines passes through the vertex of the remaining angle of the triangle.

197. If the exterior angles at A and C of the triangle $A\,B\,C$ are bisected by lines that meet in D, then

$$\text{Angle } D + \tfrac{1}{2} B = \text{one right angle.}$$

198. The angles of a triangle formed by the intersection of the lines bisecting respectively the exterior angles of a given triangle are each equal to half the sum of the adjacent angles of the given triangle; and the triangle thus formed is more nearly equilateral than the given triangle.

199. Two straight lines from the vertices of a triangle to the opposite sides cannot bisect each other.

200. A scalene triangle cannot be divided into two *equal* triangles.

201. In a quadrilateral $A\,B\,C\,D$ the diagonals $A\,C$, $D\,B$, make the angle $A\,C\,D = C\,D\,B$, and $D\,A\,C = D\,B\,C$; then $A\,B$ is parallel to $D\,C$.

202. If in a trapezoid the non-parallel sides are equal, the opposite angles are supplementary; and conversely.

203. The lines bisecting the angles of a trapezoid form a quadrilateral of whose opposite angles two are right angles and the other two supplementary.

204. $A\,B\,C$ is a triangle right-angled at B; from the point of intersection, D, of the lines bisecting the exterior angles at A and C, perpendiculars are drawn to $A\,B$ and $B\,C$, meeting $A\,B$ and $B\,C$ in E and F, respectively; then $D\,E\,B\,F$ is a square.

205. $A\,B$, $C\,D$ are parallel lines cut obliquely by $A\,E$ and perpendicularly by $A\,F$ (E, F being points in $C\,D$); $E\,G$ is drawn cutting $A\,F$ in H and $A\,B$ in G, so that $H\,G = 2\,A\,E$; then the angle $A\,E\,D = 3\,G\,E\,D$.

206. From the vertices of a parallelogram $A\,B\,C\,D$, perpendiculars are drawn meeting the diagonals in E, F, G, H, respectively; draw $E\,H$, $H\,G$, $G\,F$, $F\,E$; then $E\,H\,G\,F$ is a parallelogram equiangular to $A\,B\,C\,D$.

207. $A\,B$, $B\,C$ are two lines at right angles to each other at B; through any point D draw $A\,F$, $C\,E$ meeting $B\,C$, $A\,B$ in F, E, respectively; draw two lines from B, one bisecting $A\,F$ in G, the other bisecting $C\,E$ in H; then the angle $G\,B\,H = A\,D\,E$.

208. If from the vertices of a parallelogram perpendiculars are drawn to any straight line, the sum of the two drawn from the opposite angles is equal to the sum of the other two.

BOOK II.

130. If the two non-parallel sides of a trapezoid are produced till they meet, their point of meeting, the point of intersection of the diagonals of the trapezoid, and the middle points of the parallel sides are all in the same straight line.

131. Each of the lines which bisect the opposite sides of a quadrilateral bisects the line that joins the middle points of the diagonals of the quadrilateral.

132. The area of a triangle is equal to the product of its perimeter and the radius of the inscribed circle.

133. If in a triangle a perpendicular is drawn from the vertex to the base, the whole base is to the sum of the two sides as the difference of these sides is to the difference of the segments of the base.

134. If a, b, c, represent the sides of a triangle respectively, and $s = \frac{1}{2}(a + b + c)$, then (66, 133) the area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

135. Demonstrate Theorem XXV. by means of (I. 103) and (II. 47).

136. Discuss the second part of Theorem XXV. when the line BD, which bisects the exterior angle, is parallel to AC; also prove the proposition when BD meets CA produced (I. 34).

137. Prove (I. 176) by means of (II. 45).

138. The sum of the perpendiculars drawn from any point within a regular polygon to the several sides is constant, and is equal to as many times the apothem as the polygon has sides.

What if the point is without the polygon?

139. Two intersecting diagonals of a regular pentagon divide each other in extreme and mean ratio (V. 27).

140. The area of a triangle which has an angle of 30° is equal to one fourth the product of the sides including this angle.

141. Is there a point within a triangle such that every straight line drawn through this point bisects the triangle?

142. If a line is drawn from the vertex of an angle of a triangle perpendicular to a line bisecting another angle, and through their point of intersection a line is drawn parallel to the side opposite the first angle, this last line will bisect the other two sides.

143. If the sides of a triangle be trisected, the lines drawn through the points adjacent to each vertex, respectively, will form a triangle equal to the given triangle.

144. B and C are points in two straight lines intersecting at A; if BD is drawn perpendicular to AC, and DE perpendicular to AB; also CF perpendicular to AB, and FG to AC; then EG is parallel to BC.

145. If E is a point in the diagonal AC of the parallelogram $ABCD$, then the triangle $DAE = ABE$, and $EBC = ECD$.

146. E is a point in BC, a side of the parallelogram $ABCD$; draw DE meeting AB produced in F, and join AE and FC; then the triangle $ABE = EFC$.

147. If F is a point in the side BC (or BC produced) of the parallelogram $ABCD$, and E the intersection of the diagonals, and AF, FD are drawn, then the triangle $AFD - AED = \frac{1}{4} ABCD$.

148. A line through the intersection of the diagonals of a trapezoid parallel to its parallel sides and terminated by its non-parallel sides is bisected by the diagonals.

149. AB, CD are parallel lines, E the middle point of CD; AC, BE meet in F; AE, BD in G; then FG is parallel to AB.

150. If through E, a point in the diagonal AC of the parallelogram $ABCD$, lines are drawn parallel to AB and AD, the parallelogram $DE = EB$.

151. Through E, any point within the parallelogram $ABCD$, draw lines parallel to AD and AB; draw AE, EC, and the diagonal AC; then the difference of the parallelograms DE, EB, is twice the triangle ACE.

152. The diagonals of a trapezoid whose parallel sides are as 1 : 2 cut each other in a point of trisection (II. 103).

153. If two opposite triangles formed by the diagonals of a quadrilateral are equivalent, the bases of the other two triangles are parallel.

154. The line bisecting the diagonals of a trapezoid and terminated by its non-parallel sides is equal to half the sum of the parallel sides; and the part between the diagonals to half their difference.

155. The angle included by the perpendicular from the vertex of the right angle of a right triangle to the hypothenuse and the line from the same vertex bisecting the hypothenuse is equal to the difference of the two acute angles of the given triangle.

156. Two quadrilaterals whose diagonals are respectively equal and form equal angles are equivalent.

157. Lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

158. If lines are drawn from the vertices of a square bisecting the opposite sides in order, there will be formed a second square of one fifth the area of the first square.

159. Let $ABCD$ be a parallelogram; bisect AB and CD in E and F, and draw AF, BF, CE, DE, cutting the two diagonals of $ABCD$ in P, Q, R, S. Then $PQRS$ is a parallelogram whose area is one ninth of the area of $ABCD$.

160. When the square corner of a sheet of paper is folded so that the lines of folding are parallel and equidistant, the spaces in the several folds are as 1, 3, 5, 7, etc.

161. ABC is a triangle; AD bisects BC; BE bisects AD and meets AC in E; then $AC = 3\,AE$.

162. If in the triangle ABC, AD is drawn so as to bisect BC in D, BE to bisect AD in E, CF to bisect BE in F, and EG to bisect CF in G; then the triangle $ABC = 8\,EFG$.

163. If a quadrilateral is divided into two equivalent triangles by one diagonal, the second diagonal is bisected by the first.

164. A triangle having two sides equal to the diagonals of a quadrilateral and the included angle equal to either of the angles between these diagonals has the same area as the quadrilateral.

165. The diagonals $A\,C$, $B\,D$ of a parallelogram $A\,B\,C\,D$ intersect in E, and F is a point within the triangle $B\,C\,E$; draw $F\,B$, $F\,C$, $F\,D$, $F\,A$; then the triangle

$$A\,F\,D - B\,F\,C = B\,F\,D + A\,F\,C$$

166. The sum of the areas of two parallelograms described one on each of two sides of a triangle is equal to the area of a parallelogram described on the base of the triangle, if the altitude of this parallelogram is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two outer sides (produced) of the former parallelograms.

167. If $A\,B\,C$ is a triangle right-angled at B, and angle $B = 2\,A$, then $\overline{A\,C}^2 = 3\,\overline{B\,C}^2$.

168. If from D, the middle point of one of the sides of a triangle right-angled at C, a perpendicular $D\,E$ is drawn to the hypothenuse $A\,B$, then $\overline{E\,B}^2 - \overline{A\,E}^2 = \overline{B\,C}^2$.

169. If $A\,D$ is drawn bisecting $B\,C$ of the triangle $A\,B\,C$ right-angled at C, then $\overline{A\,B}^2 - \overline{A\,D}^2 = \frac{3}{4}\,\overline{B\,C}^2$.

170. If $B\,D$ is drawn from the vertex B to the middle point D of the base $A\,C$ of the triangle $A\,B\,C$, and $\overline{A\,B}^2 + \overline{B\,C}^2 + \overline{C\,A}^2 = 8\,\overline{B\,D}^2$, then $A\,B\,C$ is a right angle.

171. If $D\,E$ is drawn parallel to the hypothenuse $A\,C$ of the right triangle $A\,B\,C$, and $A\,E$, $D\,C$ are drawn, then $\overline{A\,C}^2 + \overline{D\,E}^2 = \overline{A\,E}^2 + \overline{D\,C}^2$.

172. If $A\,B\,C$, $A\,D\,C$ are right-angled triangles on the same side of their common hypothenuse $A\,C$, and $A\,E$, $C\,F$ are drawn perpendicular to the line (produced) joining $B\,D$, then

$$\overline{E\,B}^2+\overline{B\,F}^2=\overline{E\,D}^2+\overline{D\,F}^2$$

173. If the perpendiculars $A\,D$, $B\,E$, $C\,F$ from the vertices to the opposite sides intersect at G (I. 143), then

$$\overline{A\,B}^2-\overline{A\,C}^2=\overline{B\,G}^2-\overline{G\,C}^2$$

174. If the medial lines $A\,D$, $C\,E$ are drawn from the vertices of the acute angles A, C, of the right triangle $A\,B\,C$, then

$$4\,(\overline{A\,D}^2+\overline{C\,E}^2)=5\,\overline{A\,C}^2$$

175. $A\,C$ is the base of an isosceles triangle $A\,B\,C$, and $A\,D$ is drawn perpendicular to $B\,C$, then

$$\overline{A\,B}^2+\overline{B\,C}^2+\overline{C\,A}^2=3\,\overline{A\,D}^2+2\,\overline{B\,D}^2+\overline{C\,D}^2$$

176. $D\,E$, $D\,F$, $D\,G$ are perpendiculars from any point D to $A\,B$, $B\,C$, $C\,A$, sides of the triangle $A\,B\,C$, then

$$\overline{A\,E}^2+\overline{B\,F}^2+\overline{C\,G}^2=\overline{E\,B}^2+\overline{F\,C}^2+\overline{G\,A}^2$$

177. $A\,B\,C$ is a triangle right-angled at B; $B\,D$ is drawn perpendicular to $A\,C$, $D\,E$ to $B\,C$, $E\,D_{,}$ to $A\,C$, $D_{,}E_{,}$ to $B\,C$, $E_{,}D_{,,}$ to $A\,C$, and so on until there are n perpendiculars drawn to $A\,C$; then

$$\begin{aligned}\overline{A\,C}^2=\ &\overline{A\,D}^2+\overline{D\,D_{,}}^2+\overline{D_{,}D_{,,}}^2+\ldots+\overline{D_n\,C}^2\\ &+2\,\overline{B\,D}^2+2\,\overline{E\,D_{,}}^2+2\,\overline{E_{,}D_{,,}}^2+\ldots\ +2\,\overline{E_{n-1}\,D_n}^2.\end{aligned}$$

178. If similar polygons are described on the several sides of a right triangle as homologous sides, the sum of the two polygons on the sides is equal to the polygon on the hypothenuse.

179. On the sides, $A\,B$, $B\,C$, of a triangle right-angled at B, equilateral triangles, $A\,D\,B$, $A\,E\,C$, are described externally, and $A\,E$, $D\,C$, are joined; then the triangle $A\,D\,C+A\,E\,C=A\,D\,B\,E\,C$.

180. In the figure on page 73,

1st. If $G F$ and $K E$ are drawn, the angle
$FGA + AFG + CKE + KEC =$ a right angle.

NOTE. — Letters once written on the figure, or lines drawn, are assumed as still remaining in the subsequent propositions.

2d. The lines joining $A H$ and $C I$ are parallel.

3d. G, B, and K are in the same straight line.

4th. If perpendiculars are drawn from G and K to $A C$ produced, $B D$ is a mean proportional between them; the parts of $A C$ produced will be equal, and the perpendiculars together will be equal to $A C$.

5th. Draw $B E$. $\overline{BF}^2 - \overline{BE}^2 = \overline{AB}^2 - \overline{BC}^2$.

6th. If perpendiculars are drawn from F and E to $G A$ and $K C$, cutting them in L and M respectively, the triangles $F A L$ and $E M C$ will each be equal to $A B C$.

7th. If $F A$ and $E C$ are produced to meet $G H$ and $K I$ in N and O respectively, the triangle $A G N = C O K$.

8th. If $G H$ and $K I$ are produced to meet in P, $N P = A B$ and $P O = B C$.

9th. If $H I$ and $N O$ are drawn, the triangle $H P I = N P O$.

10th. The sum of the perpendiculars from H and I to $A C$ $= A C + 2 B D$.

11th. If the perpendiculars from H and I meet $A C$ in Q and R respectively, then $H I R Q = 2(\overline{BD}^2 + A B C)$.

12th. Draw $A K$. $B D$ passes through the intersection of $G C$ and $A K$.

13th. If $G C$ cuts $A B$ in T, and $H T$ is drawn, the triangle $A T C = T H B$.

14th. If $A K$ cuts $B C$ in S, then $B T = B S$.

15th. The triangle $A T C = A B S$.

16th. If $T D$, $D S$ are drawn, the angle $T D S$ is a right angle.

17th. $A T : T B = T B : S C$.

18th. If $T S$ is drawn, $\overline{TS}^2 = 2 A T \times S C$.

19th. $A T : S C = A D : D C = \overline{AB}^2 : \overline{BC}^2$.

20th. Polygon $F G H I K E = 4 F B E$.

21st. $A N O C$ is a square.

22d. If $G A$ and $K L$ are produced to meet in U, then $G P K U$ is a square.

23d. L, S, and I are in the same straight line.

24th. AK is perpendicular to BE; GC to FB.

25th. HT is parallel to BE.

26th. If LI is drawn, then $BILF$ is a parallelogram, and $= 2ABC$.

27th. If BF and BE cut AC in W and Y respectively, then $AW : WY = WY : YC$.

28th. $AT : SC = AW : YC = AD : DC$, and $AT : TB$, $BS : SC$, $TD : DS$ as $AB : BC$.

29th. If AK cuts BF in a, and GC cuts BE in b,
then $Aa : aS = Tb : bC$,
and $Aa : aB = Bb : bC$.

30th. $Bb = Sa$ and $Ba = Tb$.

31st. If Z is the point in which AK and GC intersect BD (12th), $WD : DZ = DZ : DY$.

32d. If WZ, ZY are drawn, WZY is a right angle, and WZ is parallel to AB, and ZY to BC.

33d. $Wa : aZ = Zb : bY$.

34th. $BZ = WY$.

35th. $AZ \times ZC = BW \times BY$.

36th. From W and Y erect perpendiculars to AC, meeting AB, BC in c, d respectively, and join cd; then $Wcd\,Y$ will be a square.

37th. From T draw a line parallel to BC, meeting AC in e, and join eS; then $TBSe$ is a square.

38th. Te passes through a, and eS through b.

39th. $Ba = Bb$ and $Sa = Tb$.

40th. The sum of the squares on the sides of the polygon $FGHIKE$ is equal to eight times the square on AC.

41st. Let AK cut BE in f, and GC cut BA in g;

$$AD : AZ = Af : AY$$
$$CD : CZ = Cg : CW$$
$$Ba : Bb = Bf : Bg = BW : BY$$
$$Za : Zg = Zb : Zf$$
$$AZ : Zf = BZ : ZD = CZ : Zg$$
$$FW : AW = WC : Wg$$
$$EY : AY = YC : Yf$$
$$GT : AT = TB : Tg$$
$$BS : SK = SC : Sf$$
$$Bg : Bf = BE : BF$$

(See also Exercises, Book III.)

BOOK III.

93. If two circumferences touch each other, the chords of each forming a straight line through the point of contact have a constant ratio.

94. If of two circles the diameter of one is the radius of the other, any chord of the first drawn from the point of contact of the two circumferences is bisected by the circumference of the second.

95. If two circumferences cut each other, the extremities of their diameters drawn from one of the points of intersection are in the same straight line with the other point of intersection.

96. If $A\,B$ and $A\,C$ are tangents to a circumference from A, and $D\,E$ is tangent to the circumference at any point between B and C, D being a point in $A\,B$, and E in $A\,C$, the perimeter of the triangle $A\,D\,E$ is constant.

97. Prove I. 144 by principles in Book III.

98. The difference between the sum of the two sides of a right triangle and the hypothenuse is equal to the diameter of the inscribed circle.

99. If lines are drawn from any point of the circumference of a circle to the vertices of an inscribed equilateral triangle, the middle line is equal to the sum of the other two.

100. The lines bisecting the angles formed by producing the opposite sides of a quadrilateral inscribed in a circle intersect at right angles.

101. The rectangle contained by two sides of a triangle is equal to the rectangle contained by the segments of the third side made by a line bisecting the opposite angle, plus the square of the bisecting line.

(Circumscribe a circle about the triangle, produce the bisecting line beyond the side to the circumference, and join the point where it meets the circumference with one extremity of this side.)

102. The rectangle contained by two sides of a triangle is equivalent to the rectangle contained by the diameter of the circumscribed circle and the perpendicular upon the third side from the vertex of the opposite angle.

103. The area of a triangle is equal to the product of its three sides divided by twice the diameter of the circumscribed circle.

104. The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by the opposite sides.

105. If a perpendicular is drawn from the vertex of a triangle $A\,B\,C$, to the base $A\,C$, the base is to the sum of the other two sides as the difference of these sides is to the difference of the segments of the base.

(With B as a centre, and the shorter of the two sides $A\,B$, $B\,C$ as a radius, describe a circle; produce $C\,B$ to meet the circumference; if the perpendicular falls without the triangle, in like manner produce $C\,A$.)

106. The diagonal and side of a square have no common measure.

107. Prove I. 143, first circumscribing a circle about the triangle and producing one of the perpendiculars to meet the circumference, by reference to III. 24.

108. Prove II. 66, first drawing a circle with the vertex of one of the acute angles as a centre, and the adjacent side as a radius, and from the vertex of the right angle drawing chords to the points where the circumference cuts the hypothenuse and the hypothenuse produced.

109. If a circle is circumscribed about a right triangle, and on each of the sides including the right angle as diameters, semicircles are described without the triangle, the sum of the areas of the crescents thus formed is equal to the area of the right triangle.

110. In the figure used in II. 180, T, B, S, D, and e are all in one circumference.

BOOK IV.

25. The difference of two lines drawn to a point in a straight line from points on opposite sides of this line is a maximum when these lines make equal angles with the given line.

26. Of parallelograms with sides mutually equal the maximum is rectangular.

27. The sum of the squares of two lines is never less than twice their rectangle.

28. The sum of the squares of the perpendiculars from a point within a rectangle to the sides is a minimum when this point is the centre of the rectangle.

29. The perimeter of an isosceles triangle is greater than that of an equal rectangle of the same altitude.

30. The sum of the triangles cut off by lines drawn from a point in the base of an isosceles triangle parallel to the sides, is the minimum when the point is at the centre of the base.

31. Of all triangles that have the same vertical angle and whose bases pass through a given point, the minimum is the one whose base is bisected at the given point.

32. Of all the squares that can be inscribed in a given square the minimum has its vertices at the middle points of the sides.

33. Of all the triangles whose vertices are on the sides of a given triangle the perimeter of the one that has its vertices at the feet of the perpendiculars from the vertices of the given triangle to the opposite sides, is the minimum.

BOOK V.

96. Between two given straight lines to draw a straight line equal to one given straight line and parallel to another.

97. To find a point at given distances from two given lines.

98. Through a given point between two straight lines to draw a line such that the part between the given lines shall be bisected at the given point.

99. To a straight line from two given points on opposite sides of it to draw lines forming an angle that is bisected by the given line.

100. From a given point to draw two straight lines, making respectively equal angles with two given intersecting straight lines.

101. Between two intersecting lines to place a given straight line so that it shall make equal angles with each.

102. In a given straight line to find a point equally distant from two given lines.

When is the problem impossible?

103. In a triangle $A B C$ to draw from a given point D, in the side $A B$, or the side produced, a straight line to $A C$ so that it shall be bisected by $B C$.

104. In the sides of a triangle to find a point from which lines drawn parallel to the other sides and limited by them are equal.

105. To draw a line parallel to one of the sides of a triangle so that the part intercepted between the other two sides shall be equal to the difference between one of these sides and the side to which the required line is parallel.

106. Draw two unequal triangles that have a side and two angles of one equal to a side and two angles of the other.

107. From a given isosceles triangle to cut off a trapezoid having for its base the base of the triangle and the other three sides equal to each other.

108. A side and two medial lines of a triangle given, to construct the triangle.

1st. When the given medial lines are from the extremities of the given side.

2d. When only one of the medial lines is from the extremity of the given side.

109. Two sides and one medial line given, to construct the triangle.

1st. When the medial line is from the vertex of the two given sides.

2d. When the medial line is not from the vertex of the two given sides.

110. The three medial lines given, to construct the triangle.

111. A side of an isosceles triangle and the sum of the perpendiculars from any point of the base to the opposite sides given, to construct the triangle.

112. The three perpendiculars from any point within to the sides of an equilateral triangle given, to construct the triangle.

113. The three lines from any point within an equilateral triangle to the vertices given, to construct the triangle.

114. The three altitudes of a triangle given, to construct the triangle.

115. The base, the sum of the sides, and the difference of the angles at the base given, to construct the triangle. (II. 68.)

116. Two angles and the sum of two sides given, to construct the triangle.

117. Two angles and the perimeter given, to construct the triangle.

118. An angle, its bisector, and the perpendicular from its vertex to the opposite side given, to construct the triangle.

119. An angle, the medial line and the perpendicular from the vertex of the given angle to the opposite side given, to construct the triangle.

120. The perpendicular, the bisector, and the medial line, from the same vertex given, to construct the triangle.

121. The feet of the perpendiculars from the vertices to the opposite sides given, to construct the triangle.

122. The middle points of the sides of a triangle given, to construct the triangle.

123. An angle, the angle between the bisector and the perpendicular from the vertex of the given angle to the opposite side, and the perpendicular given, to construct the triangle.

124. Two sides and the difference of the segments of the base made by a perpendicular from the vertical angle to the base given, to construct the triangle.

Is there any ambiguity in this Problem?

125. The base, the foot of the perpendicular from the vertex to the base, and the sum, or the difference, of the other two sides given, to construct the triangle.

126. The base and the sum of the two other sides given, to construct the triangle so that the bisector of the vertical angle shall be parallel to a given line.

127. The sum of the base and perpendicular, of the base and hypothenuse, of the perpendicular and hypothenuse of a right triangle given, to construct the triangle.

BOOK VI.

82. If straight lines are parallel the intersections of any planes passing through these lines are parallel.

83. The projections of parallel lines on any plane are parallel.

84. If parallel planes cut two planes not parallel, the angle of the intersections of one of these parallel planes with the planes not parallel is equal to the angle of the intersections of any other of the parallel planes with the planes not parallel.

85. If from a point without two lines are drawn to a plane, one perpendicular to the plane the other perpendicular to a given line in the plane, the straight line joining the feet of these perpendiculars is perpendicular to the given line.

86. $A\,O$, $B\,O$, $C\,O$ are perpendicular to each other at the common point O (40); if $A\,B$ is joined and $O\,D$ is drawn perpendicular to $A\,B$, and $C\,D$ joined, $C\,D$ is also perpendicular to $A\,B$.

87. If from two points A, A', above a plane perpendiculars $A\,B$, $A'\,B'$, are drawn to the plane, and a plane passed through A perpendicular to the line joining $A\,A'$, its line of intersection with the given plane is perpendicular to the line joining $B\,B'$.

88. If from a point in one of two intersecting planes two lines are drawn, one perpendicular to the second plane, the other perpendicular to the line of intersection of the two planes, then the plane of these two lines is perpendicular to the line of intersection of the two given planes.

89. If at the point of intersection of the perpendiculars from the vertices to the opposite sides of a triangle a perpendicular to the plane of the triangle is drawn, a line joining any vertex of the triangle to any point of the perpendicular is perpendicular to the side opposite this vertex.

90. The angle between two perpendiculars drawn from any point to two planes is equal to the angle of the planes (or to its supplement).

91. If through any point in the intersection of two planes that are perpendicular to each other two lines are drawn in one plane making equal angles with the line of intersection, then these two lines make equal angles with any line drawn in the other plane through this point.

92. If a straight line is perpendicular to a plane, its projection on any other plane will be at right angles to the intersection of the two planes.

BOOK VII.

139. If a straight line is divided into two parts, the cube of the whole is equal to the sum of the cubes of the two parts plus three times the rectangular parallelopiped whose base is their rectangle and altitude the whole line.

Or, algebraically, if a and b equal the parts,

$$(a+b)^3 = a^3 + 3\,a\,b\,(a+b) + b^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$$

140. The cube of the difference of two lines is equal to the cube of the greater minus the cube of the less and three times the rectangular parallelopiped whose base is their rectangle and altitude the difference of the lines.

Or, algebraically, if a and b are the lines,

$$(a-b)^3 = a^3 - 3\,a\,b\,(a-b) - b^3 = a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3$$

141. The sum of the squares of the twelve edges of any quadrangular prism is equal to the sum of the squares of its four diagonals plus eight times the square of the line joining the common middle points of the diagonals taken two and two.

Deduce (102) from this.

142. In a tetraedron the planes passing through the lateral edges and bisecting the edges of the base intersect in a straight line.

143. If from each of the four vertices of a tetraedron lines are drawn to the point of intersection of the medial lines of the opposite faces respectively, these lines will intersect at one point and divide each other in the ratio of 1 : 3. (This point is the centre of gravity of the tetraedron.)

144. The straight lines joining the middle points of the opposite edges of a tetraedron intersect at one point and bisect each other.

145. The plane that bisects a diedral angle of a tetraedron divides two of the faces into segments that are proportional to the faces including this diedral angle.

BOOK VIII.

65. Equal sections formed by planes cutting a sphere are equally distant from the centre of the sphere; and conversely, sections equally distant from the centre are equal.

66. Straight lines drawn from any point tangent to a sphere are equal; hence if a tetraedron is such that a sphere can be drawn so that its six edges shall be tangents to the sphere, the sum of every pair of opposite edges is the same.

67. The surface of a sphere can be divided into four, or eight, or twenty equilateral spherical triangles.

68. Any tetraedron may have a sphere inscribed in it; also one circumscribed about it.

69. Four points determine a sphere.

70. Any regular polyedron may have a sphere inscribed in it; also one circumscribed about it.

71. The six points where the extremities of three diameters, of which each is perpendicular to the plane of the other two, meet the surface of the sphere, are the six vertices, the lines joining the adjacent points the twelve edges, and the three diameters the three diagonals, of a regular octaedron.

BOOK IX.

27. If right triangles are drawn with the vertex of one of the acute angles at a given point and their bases in the same straight line, and the sum of the base and perpendicular constant, what is the locus of the vertex of the other acute angle?

28. If a square is moved so as to have the extremities of one of its diagonals upon two fixed lines at right angles to each other in the plane of the square, what is the locus of the extremities of the other diagonal?

29. What is the locus of the centre of a circle whose circumference passes through two given points?

30. What is the locus of the centre of a circle whose circumference is tangent to two given straight lines?

31. What is the locus of the centre of a circle whose circumference is tangent to a given line, straight or curved, at a given point of that line?

32. What is the locus of the centre of a circle of a given radius whose circumference passes through a given point?

33. What is the locus of the centre of a circle of a given radius whose circumference is tangent to a given straight line?

34. What is the locus of the middle point of a given line whose extremities are on two lines perpendicular to each other?

35. What is the locus of the middle points of all the chords of a given circle that pass through a given point?

36. What is the locus of the point of intersection of the perpendiculars from the vertices to the opposite sides of a triangle whose base and vertical angle are given?

37. What is the locus of the point which divides in a given ratio a line drawn from a given point within to the circumference of a given circle?